

POINCARÉ INEQUALITY

Setting: $\Omega \subseteq \mathbb{R}^d$ is a **BOUNDED** set and $p \in [1, +\infty]$.

$$\|u\|_{L^p(\Omega)} \leq C(p, d, \Omega) \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W_0^{1,p}(\Omega)$$

↑↑

NECESSARY

(think of a constant function)

NATURAL INEQUALITY: 0 at $\partial\Omega$ and

$|\nabla u|$ small
 $\Rightarrow u$ small

POINCARÉ - SOBOLEV INEQ

$\Omega \subseteq \mathbb{R}^d$ is bounded

$$\|u\|_{L^q(\Omega)} \leq C(p, q, d, \Omega) \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W_0^{1,p}(\Omega)$$

This inequality holds true for $q \in [1, p^*]$ if $p < d$.

True for every $q \in [1, +\infty)$ if $p = d$

True for every $q \in [1, +\infty]$ if $p > d$.

Proof Almost trivial if $p > d$. The case $p = d$ follows from $p < d$.

Let us consider $p < d$.

↑ [After video: c depends on $q, p^*, \text{meas}(\Omega)$]

$$\|u\|_{L^q(\Omega)} \leq C(q, \Omega) \|u\|_{L^{p^*}(\Omega)}$$

↑

Ω is bounded

$$\leq C(p, d, q, \Omega) \|\nabla u\|_{L^p(\Omega)}$$

↑

Sobolev embedding

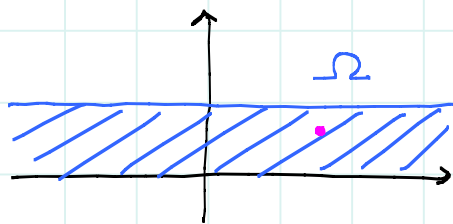
with Poincaré inequality $\rightarrow$ TRUE because $u \in W_0^{1,p}(\Omega)$

Remark In the previous proof we do NOT need any regularity on $\partial\Omega$.

THE EXAMPLE The boundedness of Ω is not really needed.

Let us consider a strip in $\mathbb{R}^2$

$$\Omega := \{(x, y) \in \mathbb{R}^2 : 0 < y < 1\}$$



I claim that $\|u\|_{L^p(\Omega)} \leq \|u_y\|_{L^p(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)}$

Proof Assume $u \in C_c^\infty(\Omega)$. Then

$$\begin{aligned} |u(x, y)| &= \left| \int_0^y u_y(x, t) dt \right| \leq \int_0^1 |u_y(x, t)| \cdot 1 dt \\ &\leq \left\{ \int_0^1 |u_y(x, t)|^p dt \right\}^{1/p} \cdot 1 \end{aligned}$$

We raise to power p and integrate in Ω

$$\begin{aligned} |u(x, y)|^p &\leq \int_0^1 |u_y(x, t)|^p dt \\ \int_{\mathbb{R}} dx \int_0^1 dy |u(x, y)|^p &\leq \int_{\mathbb{R}} dx \underbrace{\int_0^1 dy}_{=1} \int_0^1 |u_y(x, t)|^p dt = \|u_y\|_{L^p(\Omega)}^p \end{aligned}$$

If $u \in W_0^{1,p}(\Omega)$, then exploit the usual approx. argument
— 0 — 0 —

Conclusion The true assumption for Poincaré–Sobolev ineq. is that each point $x \in \Omega$ is "close enough" to some point in $\partial\Omega$, where $u = 0$.

Question In the previous inequality, can we consider different exponents in LHS and RHS? [Usual homogeneity...]

When u is NOT 0 at $\partial\Omega$

POINCARÉ - SOBOLEV - WIRTINGER

$$\|u - u_\Omega\|_{L^q(\Omega)} \leq C(d, p, q, \Omega) \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W^{1,p}(\Omega)$$

where

$$u_\Omega := \frac{1}{\text{meas}(\Omega)} \int_{\Omega} u(x) dx$$

← Average of u in Ω

True under the following assumptions

→ $\Omega \subseteq \mathbb{R}^d$ is BOUNDED

→ Ω is regular enough for Sobolev embeddings

→ $q \in [1, p^*]$ (at least if $p < d$, otherwise it is simpler)

→ Ω is CONNECTED (if not, consider $u \equiv 1$ in a ball and $u \equiv -1$ in an equal disjoint ball)

[Proof] Some initial reductions

→ we can assume that $u_\Omega = 0$ (if not, subtract u_Ω)

→ we can assume that $\|u\|_{L^q(\Omega)} = 1$ (because the ineq. is homogeneous of degree 1)

→ It is enough to prove the case $q = p$.

$$\|u\|_{L^q(\Omega)} \leq c_1(d, p, q, \Omega) \|u\|_{L^{p^*}(\Omega)}$$

↑
 Ω is bounded

$$\leq c_2(d, p, q, \Omega) \{ \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} \}$$

↑
Sobolev embedd.

$$\leq c_3(d, p, q, \Omega) \|\nabla u\|_{L^p(\Omega)}$$

↑
Ineq. for $q \Rightarrow$

The case $p=q$ Assume the inequality

$$\|u\|_{L^p(\Omega)} \leq C(d, p, \Omega) \|\nabla u\|_{L^p(\Omega)}$$

for every $u \in W^{1,p}(\Omega)$ with $\int_{\Omega} u(x) dx = 0$ and $\|u\|_{L^p(\Omega)} = 1$

is FALSE.

Then there exists a sequence $\{u_n\} \subseteq W^{1,p}(\Omega)$ s.t.

$$\|u_n\|_{L^p(\Omega)} = 1 \quad \int_{\Omega} u_n(x) dx = 0 \quad \|\nabla u_n\|_{L^p(\Omega)} \rightarrow 0$$

In particular $\|u_n\|_{1,p,\Omega}$ is bounded. Due to the compact embedding theorem, there exists

$$u_{n_k} \rightarrow u_{\infty} \text{ in } L^p(\Omega),$$

and by assumption

$$\nabla u_{n_k} \rightarrow 0 \text{ in } L^p(\Omega)$$

In the usual way we see that $\nabla u_{\infty} = 0$, and therefore u_{∞} is locally constant in Ω . But Ω is connected, and hence u_{∞} is constant.

On the other hand

$$1 = \|u_{n_k}\|_{L^p(\Omega)} \rightarrow \|u_{\infty}\|_{L^p(\Omega)}$$

} impossible if u_{∞} is constant

$$0 = \int_{\Omega} u_{n_k}(x) dx \rightarrow \int_{\Omega} u_{\infty}(x) dx$$

↑
pointwise conv. +
domination

— 0 — 0 —

Exercise P-S-W inequality is equivalent to proving that

$$\inf \left\{ \int_{\Omega} |\nabla u(x)|^p dx : u \in W^{1,p}(\Omega), \|u\|_{L^q(\Omega)} = 1, \int_{\Omega} u(x) dx = 0 \right\}$$

is strictly positive. (in this case $C = \frac{1}{\inf}$)

One could try to prove, using the DIRECT METHOD, that $\inf = \min$ (when it is min, it is strictly positive)

This proof works almost always, but there are problems in two special cases

→ $q = p^*$ (the embedding is not compact)

→ $p = 1$ (balls in L^1 are not weakly compact)

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