

Question : are there cases where $W^{1,p}(\Omega) = W_0^{1,p}(\Omega)$?

First case: TRUE if $\Omega = \mathbb{R}^d$ (Deluxe approximation)

Second example: $d=1$ and $\Omega = \mathbb{R} \setminus \{0\}$. In this case it is FALSE because

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : u(0) = 0\}$$

This is because Sobolev functions are continuous up to the boundary and if $u_n \rightarrow u$ in the full norm of Ω , then $u_n \rightarrow u$ uniformly on compact subsets of $\text{Clos}(\Omega)$

(this is true and easy if $p > 1$ because bound on full norm $\Rightarrow$ Ascoli - Arzelà; if $p=1$ it is true but less easy)

Third example : $d \geq 2$ and $\Omega = \mathbb{R}^d \setminus \{\text{origin}\}$

It is TRUE that

$$W_0^{1,p}(\Omega) = W^{1,p}(\Omega)$$

↑ TRUE in general
it what we remove
is "small"

provided that $p \leq d$.

What we need to prove: for every $u \in W^{1,p}(\Omega)$ there exists $\{u_n\} \subseteq C_c^\infty(\Omega)$ s.t. $u_n \rightarrow u$ in full norm.

Equivalently: for every $\varepsilon > 0$ there exists $u_\varepsilon \in C_c^\infty(\Omega)$ s.t.

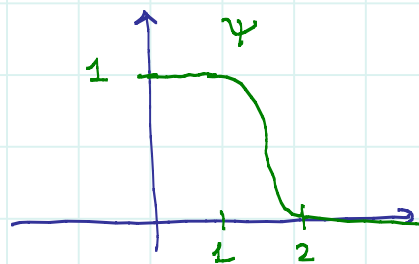
$$\|u - u_\varepsilon\|_{1,p,\Omega} \leq \varepsilon.$$

Let us prove the equivalent statement

Step 1 For every $\varepsilon > 0$ there exists $v_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ s.t.

$$\|u - v_\varepsilon\|_{1,p,\Omega} \leq \frac{1}{2} \varepsilon$$

Consider $u_n(x) := u(x) \cdot \psi\left(\frac{|x|}{n}\right)$



It is easy to see that $u_n \rightarrow u$ in full norm of $W^{1,p}(\Omega)$ (pointwise convergence + domination)

If n is large enough then

$$\|u - u_n\|_{1,p,\Omega} \leq \frac{1}{4} \varepsilon$$

↑
or $\mathbb{R}^d$

Now consider a convolution $u_n * \rho_\delta$ and if δ is small enough then

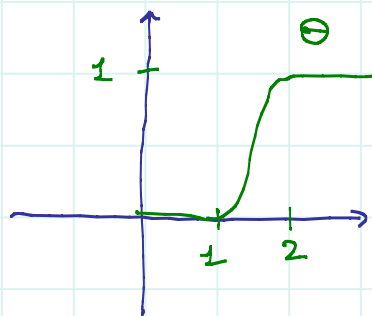
$$\|u_n - u_n * \rho_\delta\|_{1,p,\Omega} \leq \frac{1}{4} \varepsilon$$

and of course $\underbrace{u_n * \rho_\delta}_{=: v_\varepsilon} \text{ is in } C_c^\infty(\mathbb{R}^d)$

Step 2 We modify v_ε in order to obtain $u_\varepsilon \in C_c^\infty(\Omega)$ and

$$\|v_\varepsilon - u_\varepsilon\|_{1,p,\Omega} \leq \frac{1}{2} \varepsilon.$$

Define $v_n(x) := v_\varepsilon(x) \cdot \theta\left(n|x|\right)$
 ↑ coincides with v_ε if $|x| \geq \frac{2}{n}$



Claim: $v_n \rightarrow v_\varepsilon$ in full norm

$v_n \rightarrow v_\varepsilon$ in $L^p(\Omega)$ (pointwise convergence + domination)

$$\nabla v_m(x) = \nabla v_\varepsilon(x) \cdot \theta(m|x|) + v_\varepsilon(x) \cdot m \theta'(m|x|) \frac{x}{|x|}$$

$$\downarrow$$

$$\nabla v_\varepsilon(x) \text{ in } L^p$$

We need that the second term $\rightarrow 0$ in $L^p(\Omega)$

$$\int_{\Omega} \overbrace{|v_\varepsilon(x)|^p}^{\leq C} \cdot m^p \cdot \overbrace{|\theta'(m|x|)|^p}^{\leq C} dx \leq \text{const} \cdot m^p \cdot \text{meas}(B(0, \frac{2}{m}))$$

$\uparrow B(0, \frac{2}{m})$ because otherwise $\theta' = 0$

$$\sim \frac{\text{const}}{m^d}$$

$$\sim \text{const} \frac{m^p}{m^d}$$

This goes to 0 if $p < d$.

The statement is true if $p < d$.

Exercise Given $E \subset \subset \Omega \subset \subset \mathbb{R}^d$

$\uparrow$ any meas. set $\uparrow$ open set

Define

$$\text{Cap}_p(E, \Omega) := \inf \left\{ \int_{\Omega} |\nabla u(x)|^p dx : u \in W_0^{1,p}(\Omega), u(x) \geq 1 \text{ in } E \right\}$$

$\uparrow$ Capacity

$u \in C_c^\infty(\Omega) \rightarrow$ makes sense

Statement $\text{Cap}_p(\text{point}, \text{ball}) = \begin{cases} 0 & \text{if } p \leq d \\ > 0 & \text{if } p > d \end{cases}$

$\text{Cap}_p(\text{point}, \text{ball}) > 0$ if $p > d$ Claim: there exists

$$\min \left\{ \int_{\Omega} |\nabla u|^p dx : u \in W_0^{1,p}(\Omega), u(0) \geq 1 \right\}$$

$\uparrow$ ball

$\uparrow$ well defined because $p > d$

Direct method let u_n be a minimizing sequence.

$$\text{Then } \|\nabla u_n\|_{L^p(\Omega)} \leq M$$

In addition $\|u_n\|_{L^\infty(\Omega)} \leq M'$ (because $\|\nabla u_n\|$ controls the Hölder constant of u_n , which is 0 at the boundary)

A.A. $\Rightarrow$ up to subsequences $u_n \rightarrow u_\infty$ unif. in $\text{clos}(\Omega)$ and therefore also $u_\infty(0) \geq 1$.

As usual, up to further subsequences

$$\nabla u_n \rightharpoonup \nabla u_\infty \text{ weakly in } L^p(\Omega)$$

and finally

$$\liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n(x)|^p dx \geq \int_{\Omega} |\nabla u_\infty(x)|^p dx$$

$\uparrow$
convexity of the norm

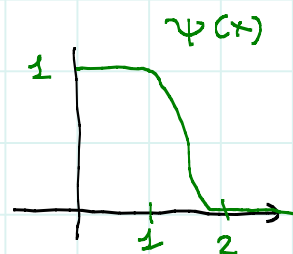
Weierstrass theorem $\Rightarrow$ the min exists.

Claim: it is > 0 . If not, then $\nabla u(x) = 0$ and this implies (true, but not trivial) that $u \equiv \text{const}$, but $u=0$ at the boundary and $u=1$ in the center.

How to prove that $\text{Cap}_p(\text{center, ball}) = 0$ if $p \leq d$

• If $p < d$, consider

$$u_n(x) = \psi(n|x|)$$



• If $p = d$, consider

$$u_n(x) = 1 - |x|^{1/n}$$



(we can modify it near 0 if we want $u_n \in C^\infty$)