

The space $W_0^{m,p}(\Omega)$

Def. Let $\Omega \subseteq \mathbb{R}^d$ be an open set, let $m \geq 1$, let $p \in [1, +\infty]$.

(1) The space $W_0^{m,p}(\Omega)$ is the closure in $W^{m,p}(\Omega)$ of $C_c^\infty(\Omega)$
(closure wrt the full norm of $W^{m,p}(\Omega)$)

(2) The space $W_0^{m,p}(\Omega)$ is the abstract completion of $C_c^\infty(\Omega)$
with respect to the metric derived from the full norm of $W^{m,p}(\Omega)$.

Easy Lemma The two definitions are equivalent.

Intuitive idea: we expect elements in $W_0^{m,p}(\Omega)$ to be Sobolev functions that vanish on $\partial\Omega$ with all derivatives up to order $m-1$.

In general this is false, but true under suitable assumptions.

Achtung! Do NOT confuse spaces such as

$$W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \quad \text{and} \quad W_0^{2,p}(\Omega)$$

For example, when $\Omega = (a,b) \subseteq \mathbb{R}$, then

$$u_m \rightarrow u \text{ and } u'_m \rightarrow u' \text{ unif.}$$

$$W_0^{2,p}((a,b)) = \{u \in W^{2,p}((a,b)) : u(a) = u'(a) = u(b) = u'(b) = 0\}$$

$$W^{2,p}((a,b)) \cap W_0^{1,p}((a,b)) = \{u \in W^{2,p}((a,b)) : u(a) = u(b) = 0\}$$

$$\text{only } u_m \rightarrow u \text{ unif.}$$

Remark If $p > d$, and $u \in W_0^{1,p}(\Omega)$, then necessarily $u = 0$ at $\partial\Omega$.

Proof Let $\{u_m\} \subseteq C_c^\infty(\Omega)$ be such that $u_m \rightarrow u$ and $\nabla u_m \rightarrow \nabla u$ in $L^p(\Omega)$.

Due to the bounds on $\|\nabla u_m\|_{L^p(\Omega)}$ and $\|u_m\|_{L^p(\Omega)}$

we can apply Ascoli - Arzelà in every ball $\subseteq \mathbb{R}^d$, and therefore $u_m \rightarrow u$ unif. on bounded sets of $\mathbb{R}^d$
 $\uparrow$
extended to 0 outside Ω .

Theorem All embedding theorems hold true in $W_0^{1,p}(\Omega)$ without assumptions on Ω and with the same p-norm inequalities as in the case of $\mathbb{R}^d$.
In particular if $u \in W_0^{1,p}(\Omega)$

(1) If $p < d$, then

$$\|u\|_{L^{p^*}(\Omega)} \leq c(p, d) \|\nabla u\|_{L^p(\Omega)}$$

(2) If $p = d$, then

$$\|u\|_{L^q(\Omega)} \leq c(p, d, q) \|u\|_{1,p,\Omega}$$

(3) If $p > d$, then

$$\|u\|_{L^\infty(\Omega)} \leq c(p, d) \|u\|_{1,p,\Omega}$$

Moreover, u is $1 - \frac{d}{p}$ Hölder cont., with a p-norm estimate on the Hölder constant.

Proof The estimates are true for every $u \in C_c^\infty(\Omega)$
 $\uparrow$
 $\mathbb{R}^d$

and then we can pass to the limit.
 $\underline{\quad} \quad \underline{\quad} \quad \underline{\quad}$

Remark Same is true for compact embeddings.

Prop. (Always true, even if Ω is "bad")

If $u \in W_0^{1,p}(\Omega)$, then $\hat{u} \in W^{1,p}(\mathbb{R}^d)$.
 $\uparrow$
extension to 0
outside Ω

Proof Let us consider $\{u_m\} \subseteq C_c^\infty(\Omega)$ such that
 $u_m \rightarrow u$ and $\nabla u_m \rightarrow \nabla u$ in $L^p(\Omega)$

Observe that actually $u_m \in C_c^\infty(\mathbb{R}^d)$ and

$$\hat{u}_m = u_m \quad \text{and} \quad \nabla \hat{u}_m = \nabla u_m$$

and in addition

$$\left. \begin{array}{l} \hat{u}_m \rightarrow \hat{u} \quad \text{in } L^p(\mathbb{R}^d) \\ \nabla \hat{u}_m \rightarrow \nabla \hat{u} \quad \text{in } L^p(\mathbb{R}^d) \end{array} \right\} \text{ outside } \Omega \text{ everything is 0.}$$

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Theorem Assume now that $\partial\Omega$ is compact and of class C^1 .
Then the following are equivalent (for a function in $W^{1,p}(\Omega)$).
(1) $u \in W_0^{1,p}(\Omega)$ $\uparrow$ [Added after video]

(2) $\hat{u} \in W^{1,p}(\mathbb{R}^d)$ extension to 0

(3) $\text{Tr } u = 0$ in $\partial\Omega$ (the trace is well-defined)

Proof (1) $\Rightarrow$ (2) is always true.

We need to show that (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (3)

(1) $\Rightarrow$ (3) Easy way: consider the definition of $\text{Tr } u$.

Extend u to $\hat{u}$. Approximate $\hat{u}$ in a delicate way. In this
 $\uparrow$
extension to 0

case we know that we can do it with $u_m \in C_c^\infty(\Omega)$.

$\text{Tr } u$ is the limit in $L^p(\partial\Omega)$ of the restrictions of u_m to the boundary, which are 0.

Alternative way Consider again the sequence $\{u_n\} \subseteq C_c^\infty(\Omega)$ that converges to u in full norm. Then exploit the continuity of the trace.

(3) $\Rightarrow$ (2) We need to prove that

$$\int_{\mathbb{R}^d} \hat{u}(x) D_{x_i} \varphi(x) dx = - \int_{\mathbb{R}^d} D_{x_i} \hat{u}(x) \varphi(x) dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^d)$$

$\uparrow$
 should be $\widehat{D_{x_i} u}(x)$

$$\begin{aligned} \text{LHS} &= \int_{\Omega} u(x) D_{x_i} \varphi(x) dx \\ \text{RHS} &= - \int_{\Omega} D_{x_i} u(x) \varphi(x) dx \end{aligned} \quad \left. \vphantom{\int_{\Omega}} \right\} \begin{array}{l} \text{They are equal if } \varphi \in C_c^\infty(\Omega) \\ \text{but this is NOT enough} \end{array} \quad \uparrow \uparrow$$

On the other hand, the integration by parts is always true

$$\int_{\Omega} u(x) D_{x_i} \varphi(x) dx = - \int_{\Omega} D_{x_i} u(x) \varphi(x) dx + \int_{\partial\Omega} \underbrace{\text{Tr } u}_{=0} \langle \vec{E}, \vec{n} \rangle d\sigma$$

$\uparrow$ $\uparrow$
 $\text{div } E(x)$ $\langle \nabla u(x), \vec{E}(x) \rangle$

$E(x) = (0, \dots, 0, \underset{\substack{\uparrow \\ i}}{\varphi}, 0, \dots, 0)$

(2) $\Rightarrow$ (1) In the usual way we reduce to the model case where $\Omega = \mathbb{R}_+^n$

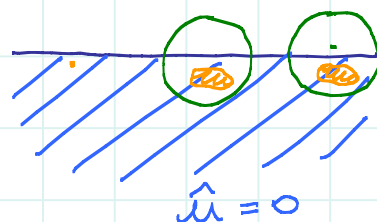
We have $\hat{u} \in W^{1,p}(\mathbb{R}_+^d)$.

We want to approximate

$$u_n \rightarrow \hat{u} \text{ in } W^{1,p}(\mathbb{R}_+^d)$$

$\uparrow$
 $\in C_c^\infty(\mathbb{R}_+^d)$

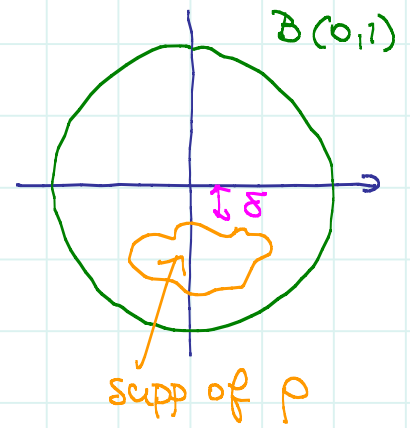
Consider as usual $u_\varepsilon := u * \rho_\varepsilon$



The support of μ_ε in general is not contained in $\mathbb{R}_+^d$

Trick to fix: consider a convolution kernel ρ

When I compute the convolution, it is equal to zero provided that the distance from the hyperplane is $\geq \varepsilon \delta$.



In this case $\mu * \rho_\varepsilon$ has compact support in $\mathbb{R}_+^d$.

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