

CONTINUITY OF THE TRACE

Trivial statement: if  $u_n \rightarrow u_\infty$  in  $W^{1,p}(\mathbb{R}_+^d)$  (in full norm) then  $\text{Tr} u_n \rightarrow \text{Tr} u_\infty$  in  $L^q(\mathbb{R}^{d-1})$  for every  $q \in [p, \hat{p}^*]$

Follows from linearity + estimates

Prop (Hölder continuity of the trace wrt to the level)

$p > 1$

For every  $u \in W^{1,p}(\mathbb{R}_+^d)$  it turns out that

$$\|u(x, y_2) - u(x, y_1)\|_{L^p(\mathbb{R}^{d-1})} \leq \left\| \frac{\partial u}{\partial y} \right\|_{L^p(\mathbb{R}_+^d)} |y_2 - y_1|^{\frac{1}{p'}} \quad \forall 0 < y_1 < y_2$$

$$\text{[Proof]} \quad |u(x, y_2) - u(x, y_1)| \leq \int_{y_1}^{y_2} \left| \frac{\partial u}{\partial y}(x, y) \right| \cdot 1 \, dy$$

$\uparrow \frac{1}{p} \qquad \qquad \qquad \uparrow \frac{1}{p'}$

$$\leq |y_2 - y_1|^{\frac{1}{p'}} \left\{ \int_{y_1}^{y_2} \left| \frac{\partial u}{\partial y}(x, y) \right|^p dy \right\}^{\frac{1}{p}}$$

We raise both sides to power  $p$  and we integrate wrt  $x$ :

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} |u(x, y_2) - u(x, y_1)|^p dx &\leq |y_2 - y_1|^{\frac{p}{p'}} \underbrace{\int_{\mathbb{R}^{d-1}} dx \int_{y_1}^{y_2} \left| \frac{\partial u}{\partial y}(x, y) \right|^p dy}_{\leq \int_{\mathbb{R}_+^d} \left| \frac{\partial u}{\partial y}(x, y) \right|^p dx dy} \\ &\leq \int_{\mathbb{R}_+^d} \left| \frac{\partial u}{\partial y}(x, y) \right|^p dx dy \end{aligned}$$

This is good if  $u \in C_c^\infty(\mathbb{R}^d)$ . Then we conclude by approx.

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Theorem Assume  $1 < p < d$   
 $\uparrow$   
 IMPORTANT

Let  $\{u_m\} \subseteq W^{1,p}(\mathbb{R}_+^d)$ . Assume that

(i)  $u_m \rightarrow u_\infty$  in  $L^p(\mathbb{R}_+^d)$

(ii) there exists  $M \in \mathbb{R}$  s.t.  $\|\nabla u_m\|_{L^p(\mathbb{R}_+^d)} \leq M$

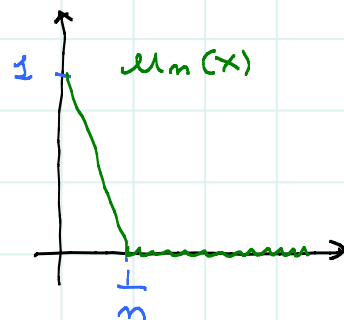
Then  $\text{Tr } u_m \rightarrow \text{Tr } u_\infty$  in  $L^q(\mathbb{R}^{d-1})$  for every  $q \in [p, \hat{p}_*)$ .  
 $\uparrow$   
 EXCLUDED

Example  $p=1$  and  $d=1$  and  $\Omega = (0,1)$

We observe that  $u_m \rightarrow 0$  in  $L^p(\Omega)$  for every  $p \in [1, +\infty)$  and

$$\|u_m\|_{L^1((0,1))} = 1 \quad \forall m \geq 1$$

$\uparrow$  only in  $L^1$



In this case  $\text{Tr } u_m \rightarrow \text{Tr } u_\infty$

General philosophy: the trace is well-defined for every  $p \geq 1$   
 but it does NOT behave well if  $p = 1$ .

Proof Step 1 It is enough to prove the case  $q = p$ . This is due to the usual interpolation inequality

conv. in  $L^p$  + bound in  $L^{\hat{p}_*} \Rightarrow$  conv. in  $L^q$  for every  $q \in [p, \hat{p}_*)$

$\uparrow$   
 follows from bound  
 on  $\|u_m\|_{1,p,\mathbb{R}_+^d}$

Step 2 Observe that  $u_\infty \in W^{1,p}(\mathbb{R}_+^d)$  and  $\|\nabla u_\infty\|_{L^p(\mathbb{R}_+^d)} \leq M$ .

Since  $\|\nabla u_m\|_p$  is bounded and  $p > 1$ , up to subsequences we can assume that

$\nabla u_m \rightarrow \text{something}$  weakly in  $L^p(\mathbb{R}_+^d)$

(Here we use weak compactness of balls in  $L^p$  when  $p > 1$ )

In the usual way we see that  $\text{something} = \nabla u_\infty$  and

$$\|\nabla u_\infty\|_{L^p(\mathbb{R}_+^d)} \leq M$$

(the norm of the gradient is convex, and therefore LSC wrt weak convergence).

Therefore the trace of  $u_\infty$  is well defined.

**Step 3** We use Morrey's trick (exchange zone)

$$\begin{aligned} \|u_\infty(x, 0) - u_m(x, 0)\|_{L^p(\mathbb{R}_+^d)} &\leq \|u_\infty(x, 0) - u_\infty(x, y)\|_{L^p} \dots \\ &\quad + \|u_\infty(x, y) - u_m(x, y)\|_{L^p} \dots \\ &\quad + \|u_m(x, y) - u_m(x, 0)\|_{L^p} \dots \end{aligned}$$

for every  $y > 0$ .

Let us raise to power  $p$  and use that

$$(A+B+C)^p \leq 3^{p-1} (A^p + B^p + C^p) \quad \forall (A, B, C) \in [0, +\infty)^3$$

$$\| \text{Tr } u_\infty - \text{Tr } u_m \|_{L^p}^p \leq 3^{p-1} \left\{ \|u_\infty(x, 0) - u_\infty(x, y)\|_{L^p}^p \right. \quad (1)$$

$$+ \|u_\infty(x, y) - u_m(x, y)\|_{L^p}^p \quad (2)$$

$$\left. + \|u_m(x, y) - u_m(x, 0)\|_{L^p}^p \right\} \quad (3)$$

Now fix  $\delta > 0$  and integrate in  $(0, \delta)$  wrt  $y$   
 $\uparrow$   
exchange zone

$$\text{LHS} = \delta \| \text{Tr } u_\infty - \text{Tr } u_n \|_{L^p}^p$$

$$\textcircled{3} \quad \int_0^\delta \underbrace{\| u_n(x, y) - u_n(x, 0) \|_{L^p}^p}_{\leq \| \frac{\partial u_n}{\partial y} \|_{L^p(\mathbb{R}_+^d)} \cdot y^{\frac{p}{p-1}}} dy \leq \underbrace{\| \frac{\partial u_n}{\partial y} \|_{L^p(\mathbb{R}_+^d)}}_{\leq M} \cdot \delta^{1+\frac{p}{p-1}}$$

$$\textcircled{4} \quad \int_0^\delta \| u_\infty(x, y) - u_\infty(x, 0) \|_{L^p}^p dy \leq M \delta^{1+\frac{p}{p-1}}$$

$$\textcircled{2} \quad \int_0^\delta dy \int_{\mathbb{R}^{d-1}} dx |u_\infty(x, y) - u_n(x, y)|^p \leq \| u_\infty - u_n \|_{L^p(\mathbb{R}_+^d)}^p$$

In conclusion we obtain

$$\delta \| \text{Tr } u_\infty - \text{Tr } u_n \|_{L^p(\mathbb{R}^{d-1})}^p \leq 3^{p-1} \left\{ 2M \delta^{1+\frac{p}{p-1}} + \| u_\infty - u_n \|_{L^p(\mathbb{R}_+^d)}^p \right\}$$

$\downarrow$   
0

We take the limsup as  $n \rightarrow +\infty$

$$\cancel{\delta} \limsup_{n \rightarrow +\infty} \| \text{Tr } u_\infty - \text{Tr } u_n \|_{L^p}^p \leq 3^{p-1} \cdot 2M \delta^{1+\frac{p}{p-1}}$$

Finally, let  $\delta \rightarrow 0^+$  and observe that  $\frac{p}{p-1} > 0$  because  $p > 1$ .

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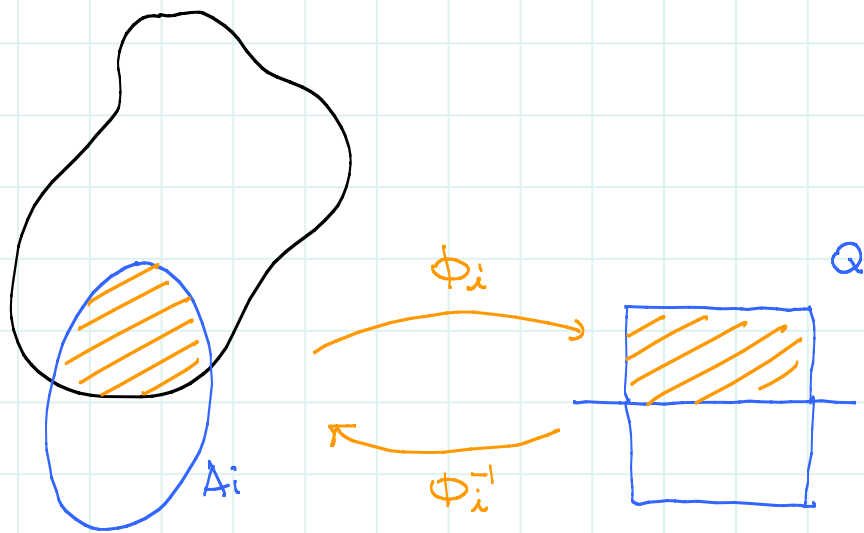
How to define the trace for a general open set

Assume  $\Omega$  is regular in the sense that  $\partial\Omega$  is compact and  $C^1$

The key point is proving inequalities such that

$$\| u \|_{L^q(\partial\Omega)} \leq C(p, d, q, \Omega) \| u \|_{L^p, \Omega}$$

For every  $u \in C_c^\infty(\mathbb{R}^d)$



→ Take any  $u \in C_c^\infty(\mathbb{R}^d)$

→ Write  $u$  as a sum  $u_0 + u_1 + \dots + u_n$  where  $u_i$  has support contained in  $A_i$

→ Move everything to  $Q$  using  $\phi_i$

→ In  $Q$  we are in the model case ( $\mathbb{R}_+^d$ )

→ Estimate the trace in  $Q$

→ Go back to  $A_i$  using  $\phi_i^{-1}$ .

The key points are that if  $u_i \in W^{1,p}(A_i)$  then  $u_i \circ \phi_i^{-1} \in W^{1,p}(\mathbb{R}^d)$  with an estimate such as

$$m \|u_i\|_{1,p,A_i} \leq \|u_i \circ \phi_i^{-1}\|_{1,p,\mathbb{R}^d} \leq M \|u_i\|_{1,p,A_i}$$

and the same is true with  $\|u_i\|_{L^q(\partial\Omega \cap A_i)}$  and  $\|u_i(x,0)\|_{L^q(\mathbb{R}^{d-1})}$

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