

TRACES OF SOBOLEV FUNCTIONS

Problem: given $u \in W^{1,p}(\Omega)$, define $u|_{\partial\Omega}$ and more generally $u|_K$ where $K \subseteq \text{Clos}(\Omega)$ is a subset

- either of codimension 1
- or of " 2.

Remark If $u \in L^p(\Omega)$ and nothing better, then there is no hope to define restrictions to sets of measure 0.

We look for a map

$$\text{Tr}: W^{1,p}(\Omega) \rightarrow \text{Space}(\partial\Omega)$$

with some reasonable properties

→ linear

→ it coincides with $u|_{\partial\Omega}$ if $u \in C^0(\text{Clos}(\Omega))$

→ it is continuous, namely if $u_n \rightarrow u_\infty$ in some sense, then

$$\text{Tr } u_n \rightarrow \text{Tr } u_\infty \text{ in } \text{Space}(\partial\Omega)$$

→ it satisfies estimates such as

$$\|\text{Tr } u\|_{\text{Space}(\partial\Omega)} \leq \text{const } \|u\|_{1,p,\Omega}$$

→ some cut. by parts formula holds true.

Easy case 1

$\Omega = (a,b) \subseteq \mathbb{R}$. In this case functions in $W^{1,p}(\Omega)$ are continuous up to the boundary

Easy case 2

$\Omega \subseteq \mathbb{R}^d$ and $u \in W^{1,p}(\Omega)$ with $p > d$ and Ω regular.

As before u is continuous (also Hölder cont.) in $\text{clos}(\Omega)$, so that in this case we can take

$$\text{Space}(\partial\Omega) = C^0(\partial\Omega)$$

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FIRST NONTRIVIAL CASE

Model case $p < d$ and Ω is a half-space

$$\Omega = \{ (x,y) \in \mathbb{R}^d : y > 0 \} = \mathbb{R}_+^d$$

$\begin{matrix} \nearrow & \nearrow \\ d-1 & 1 \end{matrix}$

$$\partial\Omega = \{ (x,y) \in \mathbb{R}^d : y = 0 \}$$

Proposition

(key estimates) let us assume that $u \in C_c^\infty(\mathbb{R}^d)$.

$$(1) \quad \|u(x,0)\|_{L^p(\mathbb{R}^{d-1})} \leq C(p,d) \|u\|_{1,p,\mathbb{R}_+^d}$$

IMPURE

u defined in $\mathbb{R}^d$
LHS is in $\mathbb{R}^{d-1}$
RHS with full norm in $\mathbb{R}_+^d$

$$(2) \quad \text{let us assume } p < d \text{ and set } \hat{p}_* := \frac{p(d-1)}{d-p}. \text{ Then}$$

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$$\|u(x,0)\|_{L^{\hat{p}_*}(\mathbb{R}^{d-1})} \leq C(p,d) \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

u in $\mathbb{R}^d$
LHS in $\mathbb{R}^{d-1}$
RHS only ∇u in $\mathbb{R}^d$

$$(3) \quad \|u(x,0)\|_{L^q(\mathbb{R}^{d-1})} \leq C(p,d,q) \|u\|_{1,p,\mathbb{R}^d} \quad \forall q \in [p, \hat{p}_*]$$

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RHS with full norm in $\mathbb{R}^d$

Proof Step 1 Consider $p=1$

$$|u(x,0)| = \left| - \int_0^{+\infty} \frac{\partial u}{\partial y}(x,y) dy \right|$$

any large R is enough

$$\leq \int_0^{+\infty} \left| \frac{\partial u}{\partial y}(x,y) \right| dy$$

We integrate wrt x :

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} |u(x,0)| dx &\leq \int_{\mathbb{R}^{d-1}} dx \int_0^{+\infty} \left| \frac{\partial u}{\partial y}(x,y) \right| dy \\ &= \int_{\mathbb{R}_+^d} \left| \frac{\partial u}{\partial y}(x,y) \right| dx dy \end{aligned}$$

Step 2 Any $p \geq 1$. We set $v(x,y) := |u(x,y)|^2 u(x,y)$ and we apply Step 1 to $v(x,y)$.

We observe that

$$\frac{\partial v}{\partial y}(x,y) = (2+1) |u(x,y)|^2 \frac{\partial u}{\partial y}(x,y)$$

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} |u(x,0)|^{2+1} dx &\leq \int_{\mathbb{R}_+^d} (2+1) |u(x,y)|^2 \left| \frac{\partial u}{\partial y}(x,y) \right| dx dy \\ \parallel &\qquad \qquad \qquad \frac{1}{p'} \qquad \qquad \qquad \frac{1}{p} \\ \|u(x,0)\|_{L^{2+1}(\mathbb{R}^{d-1})}^{2+1} &\leq (2+1) \left\| \frac{\partial u}{\partial y} \right\|_{L^p(\mathbb{R}_+^d)} \left\{ \int_{\mathbb{R}_+^d} |u(x,y)|^{2p'} dx dy \right\}^{\frac{1}{p'}} \\ &\qquad \qquad \qquad \parallel \\ &\qquad \qquad \qquad \|u\|_{L^{2p'}(\mathbb{R}_+^d)}^2 \end{aligned}$$

We proved that

$$\|u(x,0)\|_{L^{2+1}(\mathbb{R}^{d-1})}^{2+1} \leq (2+1) \left\| \frac{\partial u}{\partial y} \right\|_{L^p(\mathbb{R}_+^d)}^2 \|u\|_{L^{2p'}(\mathbb{R}_+^d)}^2$$

[After video]

Choose r such that $rp' = p \rightsquigarrow r = \frac{p}{p'} = p \frac{p-1}{p} = p-1$

In this case we obtain what we need

$$\|u(x,0)\|_{L^p(\mathbb{R}^{d-1})}^p \leq p \left\| \frac{\partial u}{\partial y} \right\|_{L^p(\mathbb{R}_+^d)} \underbrace{\|u\|_{L^p(\mathbb{R}_+^d)}^{p-1}}_{\substack{\downarrow \\ \|u\|_{L^{p,p}, \mathbb{R}_+^d}^p}}$$

Step 3 Choose r such that $rp' = p_*$

$$\frac{rp}{p-1} = \frac{pd}{d-p} \rightsquigarrow r = \frac{d(p-1)}{d-p} \rightsquigarrow r+1 = \frac{dp-d}{d-p} + 1 = \frac{dp-d+d-p}{d-p} = \frac{p(d-1)}{d-p} = \hat{p}_*$$

We obtain that

$$\|u(x,0)\|_{L^{\hat{p}_*}(\mathbb{R}^{d-1})}^{\hat{p}_*} \leq c(p,d) \left\| \frac{\partial u}{\partial y} \right\|_{L^p(\mathbb{R}_+^d)} \underbrace{\|u\|_{L^{\hat{p}_*}(\mathbb{R}_+^d)}^{\hat{p}_*-1}}_{\substack{\uparrow \\ \leq \|\nabla u\|_{L^p(\mathbb{R}^d)} \leq \|u\|_{L^{\hat{p}_*}(\mathbb{R}^d)}^{\hat{p}_*-1} \leq \hat{c}(p,d) \|\nabla u\|_{L^p(\mathbb{R}^d)}^{\hat{p}_*-1}}}$$

and the conclusion follows.

Step 4 The estimate for every $q \in [p, \hat{p}_*]$ follows by interpolation from the previous ones.

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Remark The statements for the case $p=d$ follow in the same way. (exercise!)

Theorem Let $p < d$, and let $q \in [\varphi, \hat{p}_*]$.

Then there exists

$$\text{Tr} : W^{1,p}(\mathbb{R}_+^d) \rightarrow L^q(\mathbb{R}^{d-1})$$

with the required properties

Proof $\rightarrow$ Take $u \in W^{1,p}(\mathbb{R}_+^d)$

$\rightarrow$ extend by reflection to $\hat{u} \in W^{1,p}(\mathbb{R}^d)$

$\rightarrow$ approximate deluxe $u_m \rightarrow \hat{u}$ in full norm
 $\uparrow$ in $C_c^\infty(\mathbb{R}^d)$

$\rightarrow$ observe that $\{u_m\}$ is a Cauchy sequence wrt the full norm of $W^{1,p}(\mathbb{R}^d)$

$\rightarrow$ by the Prop. we know that $\{u_m(x,0)\}$ are a Cauchy seq. in $L^q(\mathbb{R}^{d-1})$

$\rightarrow$ define $\text{Tr } u$ as the limit in $L^q(\mathbb{R}^{d-1})$ of $u_m(x,0)$.

The defin. is well posed because if $v_m \rightarrow \hat{u}$ is a different deluxe approx, then $u_1, v_1, u_2, v_2, u_3, v_3, \dots$ is again a Cauchy sequence...

If u is continuous, then $\hat{u}$ is continuous, then we can choose u_m defined by convolution and in this case

$$u_m \rightarrow \hat{u} \text{ uniformly in } \mathbb{R}^d$$

and therefore $\text{Tr } u_m \rightarrow \hat{u}(x,0)$

All the estimates and the int. by parts formula pass to the limit from u_m to u and $\text{Tr } u$.