

COMPACT EMBEDDING RESULTS

Theorem Let $\Omega \subseteq \mathbb{R}^d$, and let us assume that

(i) Ω is BOUNDED

(ii) Ω is "regular" (what is needed for Sobolev embeddings and deluxe approximation)

Then the following statements are true

$p < d$ The embedding $W^{1,p}(\Omega) \rightarrow L^q(\Omega)$ is compact for every $q \in [1, p^*)$ ↑ NOT INCLUDED

$p = d$ $W^{1,p}(\Omega) \rightarrow L^q(\Omega)$ is compact for every $q \in [1, +\infty)$

$p > d$ $W^{1,p}(\Omega) \rightarrow C^0(\text{clos}(\bar{\Omega}))$ is compact

Defn Let X and Y be metric space. A map $f: X \rightarrow Y$ is compact if the image of every bounded subset of X is relatively compact in Y . In other words

$\forall \{x_n\} \subseteq X$ bounded, $\{f(x_n)\}$ admits a converging subsequence in Y .

Proof for $p > d$ Almost trivial. If $\{u_n\} \subseteq W^{1,p}(\Omega)$ is bounded then u_n is bounded in $L^\infty(\Omega)$ and we have a control on Hölder constants. Standard A.A. $\Rightarrow$ conclusion.

Proof for $p=d$ Very easy. Consider any $q < +\infty$. Then there exists $p_1 < p=d$ such that $q < (p_1)^*$.

Observe that $\{u_m\}$ bounded in $W^{1,p}$ $\Rightarrow \{u_m\}$ bounded in W^{1,p_1} because Ω is bounded.

The conclusion follows from the case $p_1 < d$.

Proof of the true case: $p < d$ Let us start with the case $q=1$.

Let $\mathcal{U} \subseteq W^{1,p}(\Omega)$ be a bounded family. We want to prove that $\mathcal{U}$ satisfies the assumptions of the compactness thm. in $L^1(\Omega)$.

The first two assumptions are true for free. We need assumption (iii), namely

$$\int_{\mathbb{R}^d} |\hat{u}(x+\mathbb{R}) - \hat{u}(x)| dx \leq \varepsilon$$

for every $u \in \mathcal{U}$ if $|\mathbb{R}|$ is small.

Step 1 Let us set $\Omega_k := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \frac{1}{k}\}$

$$\int_{\Omega \setminus \Omega_k} |\hat{u}(x+\mathbb{R}) - \hat{u}(x)| dx \leq \int_{\Omega \setminus \Omega_k} |\hat{u}(x+\mathbb{R})| dx + \int_{\Omega \setminus \Omega_k} |\hat{u}(x)| dx$$

$$\int_{\Omega \setminus \Omega_k} |\hat{u}(x)|^{\frac{1}{p}} \cdot 1^{\frac{1}{p'}} dx \leq \left\{ \int_{\Omega \setminus \Omega_k} |\hat{u}(x)|^p dx \right\}^{\frac{1}{p}} \cdot \underbrace{\text{meas}(\Omega \setminus \Omega_k)^{\frac{1}{p'}}}_{\downarrow 0 \text{ as } k \rightarrow +\infty} \leq \|u\|_{L^p(\Omega)}$$

The same for the other term. We obtain that

$$\int_{\Omega \setminus \Omega_k} |\hat{u}(x+\mathbb{R}) - \hat{u}(x)| dx \leq 2 \|u\|_{L^p(\Omega)} \text{meas}(\Omega \setminus \Omega_k)^{\frac{1}{p'}}$$

[After video: if $p=1$ one needs to use $\frac{1}{2}$ instead of $\frac{1}{2}$]

Step 2 Assume that $|R| < \frac{1}{K}$. Assume also $u \in C_c^\infty(\mathbb{R}^d)$.
 Assume that $x \in \Omega_K$. Then

$$\begin{aligned}
 |u(x+R) - u(x)| &\stackrel{\substack{= \\ \uparrow \\ \varphi(t) := u(x+Rt)}}{=} |\varphi(1) - \varphi(0)| \\
 &\leq \int_0^1 |\varphi'(t)| dt \\
 &= \int_0^1 |\langle \nabla u(x+Rt), R \rangle| dt \\
 &\leq |R| \cdot \int_0^1 |\nabla u(x+Rt)| dt
 \end{aligned}$$

Let us integrate wrt x :

$$\begin{aligned}
 \int_{\Omega_K} |u(x+R) - u(x)| dx &\leq |R| \cdot \int_{\Omega_K} dx \int_0^1 |\nabla u(x+Rt)| dt \\
 &= |R| \int_0^1 dt \int_{\Omega_K} |\nabla u(\underbrace{x+Rt}_y)| dx \\
 &\quad \underbrace{dx = dy}_{\substack{y \\ dx = dy}} \\
 &= |R| \int_0^1 dt \int_{\substack{\Omega_K + R \\ \subseteq \Omega}} |\nabla u(y)| dy \\
 &\leq |R| \int_0^1 dt \int_{\Omega} |\nabla u(y)| dy \\
 &= |R| \int_{\Omega} |\nabla u(y)| \cdot 1 dy \\
 &\leq |R| \cdot \|\nabla u\|_{L^p(\Omega)} \cdot \text{meas}(\Omega)^{\frac{1}{p'}}
 \end{aligned}$$

Conclusion: if $u \in \mathcal{Y}$ and $|R| < \frac{1}{k}$, then

$$\int_{\Omega_k} |u(x+R) - u(x)| dx \leq |R| \cdot \|\nabla u\|_{L^p(\Omega)} \cdot \text{meas}(\Omega)^{\frac{1}{p'}}.$$

provided that also $u \in C_0^\infty(\mathbb{R}^d)$.

Why is the same true for every $u \in W^{1,p}(\Omega)$?

This is true due to deluxe approximation!!

Step 3 The goal was the following

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall u \in \mathcal{Y} \quad \forall R \in \mathbb{R}^d$$

$$|R| \leq \delta \quad \Rightarrow \quad \int_{\Omega} |\hat{u}(x+R) - \hat{u}(x)| dx \leq \varepsilon$$

We proved that

$$\int_{\Omega} |\hat{u}(x+R) - \hat{u}(x)| dx = \int_{\Omega \setminus \Omega_k} \dots + \int_{\Omega_k} \dots$$

$$\leq 2 \|u\|_{L^p(\Omega)} \cdot \text{meas}(\Omega \setminus \Omega_k)^{\frac{1}{p'}} + |R| \cdot \|\nabla u\|_{L^p} \cdot \text{meas}(\Omega)^{\frac{1}{p'}}$$

Given $\varepsilon > 0$, we choose k s.t. first term is $\leq \frac{\varepsilon}{2}$.

Then, after choosing k , we choose δ such that $\delta < \frac{1}{k}$ and the second term is $\leq \frac{\varepsilon}{2}$.

Step 4 We need compactness up to p_* .

GENERAL FACT Small / converging in L^1 + bounded in L^{p_*}
 $\Rightarrow$ small / converging in L^q for every $q \in [1, p_*)$

This is due to the interpolation inequality

$$\|u\|_{L^q(\Omega)} \leq \|u\|_{L^1(\Omega)}^\alpha \cdot \|u\|_{L^{p_*}(\Omega)}^{1-\alpha}$$

where $\frac{\alpha}{1} + \frac{1-\alpha}{p_*} = \frac{1}{q}$

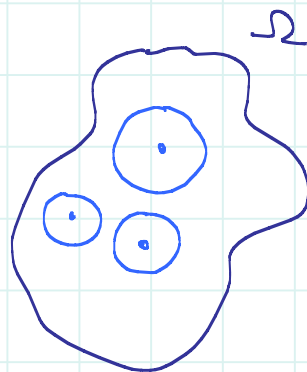
More generally

$$\|u\|_{L^q(\Omega)} \leq \|u\|_{L^{p_1}(\Omega)}^\alpha \cdot \|u\|_{L^{p_2}(\Omega)}^{1-\alpha} \quad \forall q \in [p_1, p_2]$$

where $\frac{\alpha}{p_1} + \frac{1-\alpha}{p_2} = \frac{1}{q}$ with $\alpha \in (0,1)$

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The embedding is NOT compact if $q = p_*$



Step 1 There exists a sequence of disjoint balls $B(x_n, r_n) \subseteq \Omega$

Step 2 Choose a nontrivial $\varphi \in C_c^\infty(B(0,1))$ and define

$$u_n(x) := a_n \varphi\left(\frac{x-x_n}{r_n}\right)$$

↑
choose a_n in such a way

that $\|\nabla u_n\| = 1$

↑ in $L^p(\Omega)$ or $L^p(B(x_n, r_n))$

Step 3 Observe that

$$\|u_n\|_{L^{p_*}(\Omega)} = \text{constant} \quad (\text{does NOT depend on } n)$$

↑ because of the defn. of p_*

$$\|u_n\|_{L^p(\Omega)} \rightarrow 0 \quad \text{and therefore is bounded}$$

↑ because $p < p_*$

In particular, the sequence $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$

Step 4 $\{u_n\}$ does NOT admit a converging subseq. in $L^{p^*}(\Omega)$ because

$$\int_{\Omega} |u_n(x) - u_m(x)|^{p^*} dx = \int_{\Omega} |u_n(x)|^{p^*} dx + \int_{\Omega} |u_m(x)|^{p^*} dx$$

$\uparrow$
balls are disjoint

$$= 2 \cdot \text{constant}$$

$\Rightarrow \{u_n\}$ has no Cauchy subseq.

Remark $u_n \rightarrow 0$ for every $p \in [1, p^*)$

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