

Compactness results

Basic idea: assume that $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$. We want to show that some subseq. converges in some STRONG sense (weak conv. is trivial)

Compactness in metric spaces

Let X be a metric space. Then the following are equivalent

- (i) X is covering compact
- (ii) X is sequentially compact
- (iii) X is complete and TOTALLY BOUNDED

Def. A metric space (X, d) is tot. bounded if for every $\varepsilon > 0$ there exists a FINITE subset $S \subseteq X$ such that

↑ ε -net

$$X \subseteq \bigcup_{s \in S} B(s, \varepsilon)$$

Prop. (relatively compact subsets)

Let $Y \subseteq X$ be a subset of a COMPLETE metric space X .

Then the following are equivalent:

- (i) $\text{Clos}(Y)$ is compact (wrt the restriction of the metric)
- (ii) for every $\{y_n\} \subseteq Y$ there exists $y_n \rightarrow y_\infty$ with y_∞ not necessarily in Y
- (iii) Y is totally bounded (wrt the restriction of the metric).

Defn $Y \subseteq X$ is relatively compact if it satisfies any of (i), (ii), (iii)
↑ complete

Prop. Let X be a metric space, and let $Y \subseteq X$ be a subset

Then the following are equivalent

(i) Y is totally bounded, namely

$$\forall \varepsilon > 0 \exists \text{ FINITE } S \subseteq Y \text{ s.t. } Y \subseteq \bigcup_{y \in S} B(y, \varepsilon)$$

(ii) $\forall \varepsilon > 0 \exists K_\varepsilon \subseteq X$ s.t. K_ε is totally bounded (for example K_ε compact)
and

$$Y \subseteq \bigcup_{y \in K_\varepsilon} B(y, \varepsilon)$$

"Existence of a compact ε -net $\Rightarrow$ existence of finite ε -net"

Proof of (ii) $\Rightarrow$ (i) Given $\varepsilon > 0$ find K_ε s.t.

$$Y \subseteq \bigcup_{y \in K_\varepsilon} B(y, \frac{\varepsilon}{2})$$

then find $S \subseteq K_\varepsilon$ s.t. $K_\varepsilon \subseteq \bigcup_{y \in S} B(y, \frac{\varepsilon}{2})$. Conclude that

$$Y \subseteq \bigcup_{y \in S} B(y, \varepsilon)$$

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Conclusion of the Summary Assume that X is a complete metric space. Assume that

$$\forall \varepsilon > 0 \exists K_\varepsilon \subseteq X \text{ compact s.t. } Y \subseteq \bigcup_{y \in K_\varepsilon} B(y, \varepsilon)$$

$\uparrow$
tot. bounded is enough

Then Y is relatively compact.

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Theorem (Compactness in L^p) Let us assume that $\Omega \subseteq \mathbb{R}^d$ and $\mathcal{Y} \subseteq L^p(\Omega)$.

Let us assume that

(i) Ω is BOUNDED

(ii) $\mathcal{Y}$ is bounded in $L^p(\Omega)$, namely

$$\exists M \in \mathbb{R} \quad \forall f \in \mathcal{Y} \quad \|f\|_{L^p(\Omega)} \leq M$$

(iii) $\mathcal{Y}$ is "equicontinuous in $L^p(\Omega)$ ", namely

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall f \in \mathcal{Y} \quad \forall R \in \mathbb{R}^d$$

$$|R| \leq \delta \quad \Rightarrow \quad \int_{\mathbb{R}^d} |\hat{f}(x+R) - \hat{f}(x)|^p dx \leq \varepsilon^p$$

[After video: Ω is enough]
↑ extension of u to $\mathbb{R}^d$ outside Ω

$$\|\tau_R \hat{f} - \hat{f}\|_{L^p(\mathbb{R}^d)} \leq \varepsilon$$

↑ R -translation of u .
↑ [Ω is enough]

[Some philosophy in the VIDEO]

Then $\mathcal{Y}$ is relatively compact in $L^p(\Omega)$.

Proof We need to show that

$$\forall \varepsilon > 0 \quad \exists K_\varepsilon \text{ s.t. } \mathcal{Y} \subseteq \bigcup_{g \in K_\varepsilon} B(g, \varepsilon)$$

↑ compact

$$\text{namely } \forall f \in \mathcal{Y} \quad \exists g \in K_\varepsilon \text{ s.t. } \|f - g\|_{L^p(\Omega)} \leq \varepsilon. \quad (*)$$

Consider $\delta > 0$ corresponding to ε in (iii), and define

$$K_\varepsilon := \{ f * \rho_\delta : f \in \mathcal{Y} \}$$

↑ convolution

Step 1 Let us check (*). Natural choice : $g = f * \rho_\delta$

$$f(x) - g(x) = f(x) - \int_{\mathbb{R}^d} f(x + \delta y) \rho(y) dy$$

$$= \int_{\mathbb{R}^d} f(x) \rho(y) dy - \int_{\mathbb{R}^d} f(x + \delta y) \rho(y) dy$$

$$= \int_{\mathbb{R}^d} (f(x) - f(x + \delta y)) \rho(y) dy$$

$$|f(x) - g(x)| \leq \int_{\mathbb{R}^d} |f(x + \delta y) - f(x)| \rho(y) dy \quad \frac{1}{p} + \frac{1}{p'} = 1$$

$$= \int_{\mathbb{R}^d} \underbrace{|f(x + \delta y) - f(x)| \rho(y)}^{\frac{1}{p}} \underbrace{\rho(y)}^{\frac{1}{p'}} dy$$

$$\leq \left\{ \int_{\mathbb{R}^d} |f(x + \delta y) - f(x)|^p \rho(y) dy \right\}^{\frac{1}{p}} \underbrace{\left\{ \int_{\mathbb{R}^d} \rho(y) dy \right\}^{\frac{1}{p'}}}_{=1}$$

$$|f(x) - g(x)|^p \leq \int_{\substack{\mathbb{R}^d \\ \uparrow B(0,1)}} |f(x + \delta y) - f(x)|^p \rho(y) dy$$

Integrate over Ω

$$\int_{\Omega} |f(x) - g(x)|^p dx \leq \int_{\substack{\mathbb{R}^d \\ \uparrow B(0,1)}} dy \rho(y) \int_{\substack{\Omega \\ \uparrow \text{with } |x| \leq \delta}} |f(x + \delta y) - f(x)|^p dx$$

$\underbrace{\qquad\qquad\qquad}_{=1} \leq \varepsilon^p$
 $\leq \varepsilon^p$

"Some f are actually $\hat{f}$ ".

Step 2 Elements of K_ε are bounded in the L^∞ sense.

$$|(f * \rho_\varepsilon)(x)| \leq \int_{\substack{\mathbb{R}^d \\ \uparrow \\ B(0,1)}} |\hat{f}(x+\varepsilon y)| \cdot |\rho(y)| dy$$

$\frac{1}{p} \qquad \frac{1}{p'}$

$$\leq \left\{ \int_{\mathbb{R}^d} |\hat{f}(x+\varepsilon y)|^p dy \right\}^{1/p} \left\{ \int_{\mathbb{R}^d} \rho(y)^{p'} dy \right\}^{1/p'}$$

$\begin{matrix} \text{"} \\ z \end{matrix}$
 $\varepsilon^d dy = dz$

$\uparrow$
fixed number

$$\left\{ \int_{\mathbb{R}^d} |\hat{f}(z)|^p \frac{1}{\varepsilon^d} dz \right\}^{1/p}$$

$\uparrow$
bounded

Step 3 Elements of K_ε are Lipschitz continuous with a common bound on the Lip. constant.

Since elements of K_ε are of class C^∞ , it is enough to bound the gradient

$$\begin{aligned} \nabla (f * \rho_\varepsilon)(x) &= \nabla \left(\int_{\mathbb{R}^d} \hat{f}(x+\varepsilon y) \rho(y) dy \right) \\ &= \nabla \left(\frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} \hat{f}(z) \rho\left(\frac{z-x}{\varepsilon}\right) dz \right) \\ &= - \frac{1}{\varepsilon^{d+1}} \int_{\mathbb{R}^d} \hat{f}(z) \nabla \rho\left(\frac{z-x}{\varepsilon}\right) dz \end{aligned}$$

$$\begin{aligned} |\nabla (f * \rho_\varepsilon)(x)| &\leq \frac{1}{\varepsilon^{d+1}} \int_{\mathbb{R}^d} |\hat{f}(z)| \cdot |\nabla \rho\left(\frac{z-x}{\varepsilon}\right)| dz \\ &\leq \frac{1}{\varepsilon^{d+1}} \|\hat{f}\|_{L^p(\Omega)} \left\{ \int_{\mathbb{R}^d} |\nabla \rho\left(\frac{z-x}{\varepsilon}\right)|^{p'} dz \right\}^{1/p'} \end{aligned}$$

$\frac{1}{p} \qquad \frac{1}{p'}$

number

Conclusion : the set K_ε satisfies the assumptions of the standard Ascoli - Arzelà in $C(\text{clos}(\Omega))$ which is bounded and compact.

Therefore K_ε is relatively compact wrt unif. convergence, and therefore also with respect to L^p convergence.

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