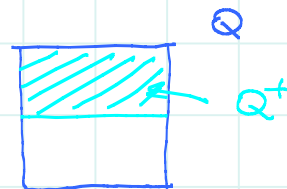


EXTENDER FOR SMOOTH OPEN SET

Let us consider the standard "cubes"

$$Q := \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} : \|x\| < 1, |y| < 1\}$$



$$Q^+ := \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} : \|x\| < 1, 0 < y < 1\}$$

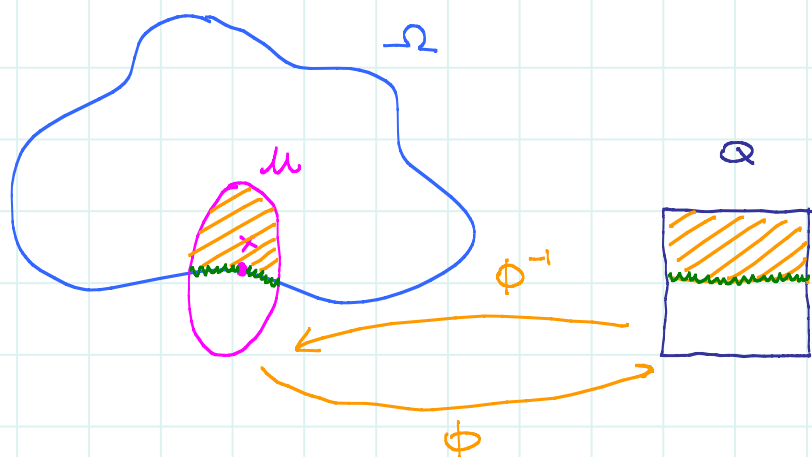
Defn Let us assume that $\Omega \subseteq \mathbb{R}^d$ is an open set with the following properties

- $\partial\Omega$ is compact
- for every $x \in \partial\Omega$ there exists an open set $U \ni x$ and a function

$$\Phi: U \rightarrow Q$$

such that

- Φ is invertible
- Φ and Φ^{-1} are of class C^1
- J_Φ and $J_{\Phi^{-1}}$ are bounded matrices
- $\Phi(U \cap \Omega) = Q^+$
- $\Phi(U \cap \partial\Omega) = Q_0 = \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} : \|x\| < 1, y = 0\}$



Partition of the unity of type B

Let $\Omega \subset \mathbb{R}^d$ be an open set with $\partial\Omega$ compact.

Assume that $A_1, \dots, A_m$ is a FINITE open covering of $\partial\Omega$ by bounded open set.

Let us set $A_0 := \Omega$.

Then there exist functions $\psi_0, \psi_1, \dots, \psi_m$ such that

(i) $\text{supp}(\psi_i) \subseteq A_i$ for $i=0, 1, \dots, m$

(this means compact support for $i \geq 1$ but not necessarily for $i=0$ if Ω is not bounded)

(ii) $\psi_i(x) \geq 0$ for every $x \in \mathbb{R}^d$ and every $i=0, 1, \dots, m$.

(iii) as expected

$$\sum_{i=0}^m \psi_i(x) = 1 \quad \forall x \in \text{Clos}(\Omega)$$

(and actually in an open set that contains $\text{Clos}(\Omega)$).

Second ingredient (Composition with Sobolev outside)

Assume $\phi: A \rightarrow B$ has the properties stated above
 $\uparrow$ open sets $\uparrow$ ($J\phi$ and $J\phi^{-1}$ are bounded)

Then for every $v \in W^{1,p}(B)$ it turns out that $u = v \circ \phi \in W^{1,p}(A)$ and we have a control on the full norms

$$m \|v\|_{1,p,B} \leq \|u\|_{1,p,A} \leq M \|v\|_{1,p,B}$$

The same using ϕ^{-1} instead of ϕ .

Theorem

Let us assume that Ω is as in the previous definition.

Then Ω admits a $(1, p)$ -extender

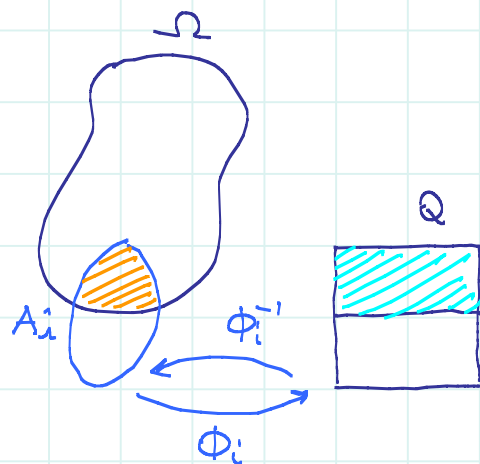
Proof

Let $A_1, \dots, A_n$ be a finite covering of $\partial\Omega$ with open sets as in the definition.

For every $i = 1, \dots, n$, let

$$\Phi_i : A_i \rightarrow Q$$

the corresponding map.



Let $\psi_0, \psi_1, \dots, \psi_n$ be the corresponding partition of the unity of type B. Let us set

$$\mu_i(x) = \mu(x) \psi_i(x) \quad \forall x \in \Omega$$

so that

$$\mu(x) = \sum_{i=0}^n \mu_i(x)$$

$\uparrow$ has support $\subseteq A_i$

For every $i = 1, \dots, n$ we set

$$\nu_i(x) := \mu_i(\Phi_i^{-1}(x)) \quad \forall x \in Q^+$$

$$\tilde{\nu}_i(x) := E_{\text{even}} \nu_i(x) \quad \forall x \in Q \leftarrow \text{FULL CUBE}$$

$$\hat{\mu}_i(x) := \tilde{\nu}_i(\Phi_i(x)) \quad \forall x \in A_i$$

and finally

$$\hat{\mu}(x) := \mu_0(x) + \sum_{i=1}^n \hat{\mu}_i(x) \quad \forall x \in \bigcup_{i=0}^n A_i$$

open neighborhood
of $\text{clos}(\Omega)$
 $\uparrow$

The claim is that $u \rightarrow \hat{u}$ is a $(1,p)$ extender

- It is linear (almost trivial)
- It is an extension. To this end, it is enough to show that

$$\hat{u}_i(x) = u_i(x) \quad \forall x \in A_i \cap \Omega$$

(if $x \in A_i \cap \Omega$ we did nothing)

- We need the norm control. It is enough to show that

$$\|\hat{u}_i\|_{1,p,A_i} \leq M_i \|u_i\|_{1,p,A_i \cap \Omega}$$

and this is true because

$$\|\hat{u}_i\|_{1,p,A_i} \leq \text{const} \cdot \|\hat{v}_i\|_{1,p,Q}$$

$\uparrow$
2nd ineq.

$$\leq 2 \cdot \text{const} \cdot \|v_i\|_{1,p,Q^+}$$

$$\leq \text{const} \cdot 2 \cdot \text{const} \cdot \|u_i\|_{1,p,A_i \cap \Omega}$$

The conclusion follows because

$$\|\hat{u}\|_{1,p,\mathbb{R}^d} \leq \|u_0\|_{1,p,\mathbb{R}^d} + \sum_{i=1}^n \|\hat{u}_i\|_{1,p,\mathbb{R}^d}$$

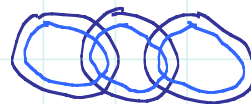
$\uparrow \quad \quad \quad \uparrow$
 $\Omega \quad \quad \quad A_i$

$$\leq \|u_0\|_{1,p,\Omega} + M \sum_{i=1}^n \|u_i\|_{1,p,A_i \cap \Omega}$$

$$\leq \text{const} \|u\|_{1,p,\Omega}$$

— 0 — 0 —

Proof of partition of the unity



Step 1 We replace $A_0, A_1, \dots, A_n$ by smaller open sets defined as

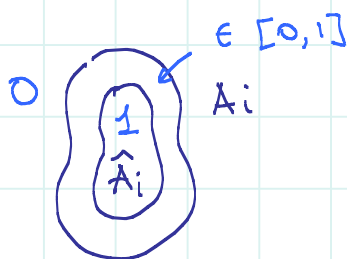
$$A_{i,k} := \{x \in A_i : \text{dist}(x, \partial A_i) > \frac{1}{k}\}$$

We show that, if k is large enough, then $\{A_{i,k}\}_{i=0,1,\dots,n}$ are again an open covering of $\text{Clos}(\Omega)$.

We call these new sets

$$\hat{A}_0, \hat{A}_1, \dots, \hat{A}_n$$

Step 2 For every $i=0,1,\dots,n$ we find θ_i which is



Step 3 We set

$$\psi_0(x) := \theta_0(x)$$

$$\psi_1(x) := \theta_1(x) (1 - \theta_0(x))$$

$$\psi_2(x) := \theta_2(x) (1 - \theta_1(x)) (1 - \theta_0(x))$$

$\vdots$

$$\psi_k(x) := \theta_k(x) \prod_{i=0}^{k-1} (1 - \theta_i(x))$$

This works! We only need to check that $\sum_{i=0}^n \psi_i(x) = 1$ for every $x \in \bigcup_{i=0}^n \hat{A}_i$. This follows from

$$1 - \sum_{i=0}^n \psi_i(x) = \prod_{i=0}^n (1 - \theta_i(x)) \quad \forall x \in \mathbb{R}^d$$

[This equality with k instead of n can be proved by induction]