

EXTENDERS — MODEL CASE

Why do we need extenders? They are a suff. cond. for
 $\rightarrow$ deluxe approx. in $\Omega \subseteq \mathbb{R}^d$
 $\rightarrow$ Sobolev embeddings in $\Omega \subseteq \mathbb{R}^d$

Model case

$A \subseteq \mathbb{R}^{d-1}$ open set

$$C_A := \{ (x, y) \in \mathbb{R}^d : x \in A, y \in (-1, 1) \} \quad \text{cylinder}$$

$\downarrow -1$ $\uparrow 1$ $\nwarrow \in \mathbb{R}$
 $y \in \mathbb{R}$

$$C_A^+ := \{ (x, y) \in \mathbb{R}^d : x \in A, y \in (0, 1) \}$$

We can consider also the "infinite cylinders"

$$C_A := A \times \mathbb{R}$$

$$C_A^+ := A \times (0, +\infty)$$

Goal: given $u \in W^{1,p}(C_A^+)$, find $\hat{u} \in W^{1,p}(C_A)$ that extends u .

Def. For every $u \in W^{1,p}(C_A^+)$, let us consider

$$[E_{\text{even}} u](x, y) := \begin{cases} u(x, y) & \text{if } x \in A \text{ and } y \in (0, 1) \\ u(x, -y) & \text{if } x \in A \text{ and } y \in (-1, 0) \end{cases}$$

$$[E_{\text{odd}} u](x, y) := \begin{cases} u(x, y) & \text{as before} \\ -u(x, -y) & \text{as before} \end{cases}$$

Theorem Let $u \in W^{1,p}(C_A^+)$. Then $\hat{u} := [E_{\text{even}} u](x) \in W^{1,p}(C_A)$,
and

$$D_{x_i} \hat{u} = E_{\text{even}} D_{x_i} u \quad \text{if } i \in \{1, \dots, d-1\}$$

$$D_y \hat{u} = E_{\text{odd}} D_y u$$

In addition $\|\hat{u}\|_{1,p,C_A} = 2 \|u\|_{1,p,C_A}$

Proof It is trivial that E_{even} is linear and extends u .
Let us consider derivatives.

$D_{x_i} \hat{u}$ with $i=1, \dots, d-1$ We need to prove that

$$\int_{C_A} \hat{u}(x,y) D_{x_i} \varphi(x,y) dx dy = - \int_{C_A} E_{\text{even}} D_{x_i} u(x,y) \varphi(x,y) dx dy$$

for every $\varphi \in C_c^\infty(C_A)$.

$$\text{LHS} = \int_A dx \int_0^1 dy u(x,y) D_{x_i} \varphi(x,y) + \int_A dx \int_{-1}^0 dy u(x,-y) D_{x_i} \varphi(x,y)$$

$$= \int_A dx \int_0^1 dy u(x,y) D_{x_i} \varphi(x,y) + \int_A dx \int_0^1 dy u(x,y) D_{x_i} \varphi(x,-y)$$

$$= \int_{C_A^+} u(x,y) \{ D_{x_i} \varphi(x,y) + D_{x_i} \varphi(x,-y) \} dx dy$$

$$= \int_{C_A^+} u(x,y) D_{x_i} \psi(x,y) dx dy \quad \text{where } \psi(x,y) := \varphi(x,y) + \varphi(x,-y)$$

If we do the same with the RHS we find

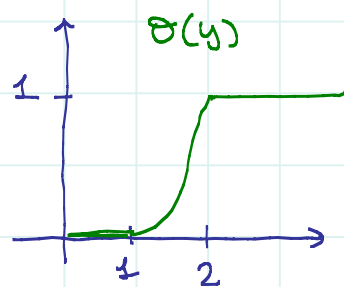
$$\text{RHS} = - \int_{C_A^+} D_{x_i} u(x, y) \psi(x, y) dx dy \quad \text{with } \psi \text{ as above}$$

So we reduced ourselves to proving that

$$\int_{C_A^+} u(x, y) D_{x_i} \psi(x, y) dx dy = - \int_{C_A^+} D_{x_i} u(x, y) \psi(x, y) dx dy$$

This would be trivial if $\psi \in C_c^\infty(C_A^+)$, but in general this is not true.

Consider a cutoff function θ as in the figure.



Consider

$$\psi_n(x, y) := \psi(x, y) \cdot \theta(ny)$$

↑
coincides with ψ if $y \geq \frac{2}{n}$
and vanishes if $y \leq \frac{1}{n}$

In particular, $\psi_n \in C_c^\infty(C_A^+)$ and therefore

$$\int_{C_A^+} u(x, y) D_{x_i} \psi_n(x, y) dx dy = - \int_{C_A^+} D_{x_i} u(x, y) \psi_n(x, y) dx dy$$

is true for every $n \geq 1$. Now we pass to the limit.

In the RHS we have pointwise convergence and domination

$$|D_{x_i} u(x, y) \psi_n(x, y)| \leq \text{constant } |D_{x_i} u(x, y)| \in L^p$$

The LHS is equal to

$$\int_{\mathbb{C}_A^+} u(x,y) \Theta(ny) D_x \psi(x,y) dx dy$$

and we conclude in the same way.

$D_y \hat{u}$ We need to prove that

$$\int_{\mathbb{C}_A} \hat{u}(x,y) D_y \varphi(x,y) dx dy = - \int_{\mathbb{C}_A} E_{\text{odd}} D_y u(x,y) \varphi(x,y) dx dy$$

for every $\varphi \in C_c^\infty(\mathbb{C}_A)$.

With the same variable change we find that

$$\text{LHS} = \int_{\mathbb{C}_A^+} u(x,y) D_y \psi(x,y) dx dy$$

$$\text{RHS} = - \int_{\mathbb{C}_A^+} D_y u(x,y) \psi(x,y) dx dy$$

$$\text{but now } \psi(x,y) := \varphi(x,y) \overset{\uparrow\uparrow}{-} \varphi(x,-y)$$

We introduce $\psi_n(x,y) := \psi(x,y) \cdot \Theta(ny)$ as before, and we observe that, for every $n \geq 1$,

$$\begin{aligned} \int_{\mathbb{C}_A^+} u(x,y) D_y \psi_n(x,y) dx dy &= - \int_{\mathbb{C}_A^+} D_y u(x,y) \underbrace{\psi_n(x,y)}_{\psi(x,y)} dx dy \\ &\quad - \int_{\mathbb{C}_A^+} D_y u(x,y) \psi(x,y) dx dy \end{aligned}$$

$$\text{LHS} = \int_{\mathbb{C}_A^+} u(x,y) \Theta(ny) D_y \psi(x,y) dx dy + \int_{\mathbb{C}_A^+} u(x,y) \psi(x,y) \overset{\uparrow\uparrow}{n \Theta'(ny)} dx dy$$

The first term $\leadsto$ no problem

As for the second term we observe that

- $n |\theta'(y)| \leq \text{const.} \cdot n$ and vanishes when $y \geq \frac{2}{n}$
- $\psi(x, 0) = 0$ for every $x \in A$ and in addition ψ is Lip. cont. and therefore

$$|\psi(x, y)| = |\psi(x, y) - \psi(x, 0)| \leq L y$$

As a consequence

$$\begin{aligned} \int_{C_A^+} |u(x, y) \psi(x, y) n \theta'(ny)| dx dy \\ \leq \int_A dx \int_0^{\frac{2}{n}} dy |u(x, y)| \cdot L y \cdot n \cdot \text{const} \\ \leq \int_A dx \int_0^{\frac{2}{n}} dy \text{const} \cdot L \cdot |u(x, y)| \xrightarrow{\substack{\uparrow \\ \text{dominated} \\ \text{convergence}}} 0 \end{aligned}$$

— 0 — 0 —

Remark In general it is NOT true that

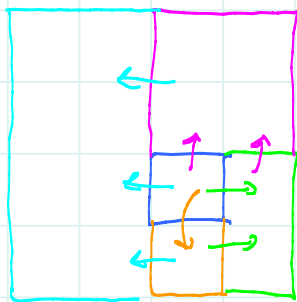
$$\int_{C_A^+} u(x, y) D_{x_i} \psi(x, y) dx dy = - \int_{C_A^+} D_{x_i} u(x, y) \psi(x, y) dx dy$$

if ψ has not compact support in C_A^+ .

One has to consider the boundary terms when $y=0$

— 0 — 0 —

Extender for squares / rectangles



→ In four steps we extend by reflection to a bigger square

→ Then we multiply by a function of class C^∞ that is

- 1 in the smaller square
- 0 outside the big square

— 0 — 0 —