

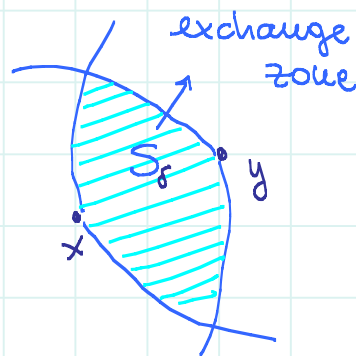
Sobolev embedding with $p > d$

$u \in W^{1,p}(\mathbb{R}^d) \Rightarrow u \in L^\infty(\mathbb{R}^d)$ with impure estimate and u is $1 - \frac{d}{p}$ Hölder cont. with pure estimate of the Hölder constant

Step 1 Let us assume that $u \in C_c^\infty(\mathbb{R}^d)$.

MORREY TRICK (exchange zone)

$$|u(y) - u(x)| \leq |u(y) - u(z)| + |u(z) - u(x)|$$



We set $\delta := |y - x|$ and we integrate in S_δ

$$\begin{aligned} \text{LHS} &= \int_{S_\delta} |u(y) - u(x)| dz = |u(y) - u(x)| \cdot \text{meas}(S_\delta) \\ &= |u(y) - u(x)| \cdot C(d) \delta^d \end{aligned}$$

For the RHS, we have the following estimate

$$\begin{aligned} |u(z) - u(x)| &= |\varphi(1) - \varphi(0)| = \left| \int_0^1 \varphi'(t) dt \right| \\ &\quad \uparrow \\ &\quad \varphi(t) = u(x + t(z-x)) \\ &\leq \int_0^1 |\varphi'(t)| dt \\ &= \int_0^1 |\langle \nabla u(x + t(z-x)), z-x \rangle| dt \\ &\leq \underbrace{|z-x|}_{\leq \delta} \cdot \int_0^1 |\nabla u(x + t(z-x))| dt \leq \delta \int_0^1 |\nabla u(x + t(z-x))| dt \end{aligned}$$

Now we integrate

$$\begin{aligned} \int_{S_\delta} |u(z) - u(x)| dz &\leq \delta \int_{S_\delta} dz \int_0^1 |\nabla u(x + t(z-x))| dt \\ &= \delta \int_0^1 dt \int_{S_\delta} |\nabla u(\underbrace{x + t(z-x)}_w)| dz \\ &\quad dw = t^d dz \end{aligned}$$

$$\leq \delta \int_0^1 t^{-d} dt \int_{B(x, \delta t)} |\nabla u(w)| \cdot 1 dw$$

$\frac{1}{p} \quad \frac{p-1}{p}$

where lies w ? $|w-x| = t|z-x| \leq \delta t \Rightarrow w \in B(x, \delta t)$

$$\leq \delta \int_0^1 t^{-d} dt \left\{ \int_{B(x, \delta t)} |\nabla u(w)|^p dw \right\}^{\frac{1}{p}} \cdot \text{meas}(B(x, \delta t))^{\frac{p-1}{p}}$$

$\leq \|\nabla u\|_{L^p(\mathbb{R}^d)} \quad C(d, p) t^{\frac{d}{p}} \delta^{\frac{p-1}{p}}$

$$\leq C(d, p) \delta^{1+d-\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d)} \int_0^1 t^{d+d-\frac{d}{p}} dt$$

$$= C_1(d, p) \delta^{1+d-\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d)} \int_0^1 \frac{1}{t^{\frac{d}{p}}} dt$$

$$\int_0^1 \frac{1}{t^{\frac{d}{p}}} dt$$

this is a number
because $\frac{d}{p} < 1$

At the end of the day we find that

$$|u(x) - u(y)| C(d) \delta^{\frac{d}{p}} \leq 2 C_1(d, p) \delta^{\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

$\uparrow$
 $|x-y|$

$1+d-\frac{d}{p}$

Hölder exponent that
we expected

Step 2 Again $\mu \in C_c^\infty(\mathbb{R}^d)$. We want the $L^\infty(\mathbb{R}^d)$ estimate

Let us consider any $x \in \mathbb{R}^d$. For sure

$$\int_{B(x, 24)} |\mu(y)|^p dy \leq \int_{\mathbb{R}^d} |\mu(y)|^p dy$$

" " $\|\mu\|_{L^p(\mathbb{R}^d)}^p$

$$|\mu(\hat{y})|^p \text{ means } (B(x, 24))$$

↑
mysterious point
in the Ball

$$\Rightarrow |\mu(\hat{y})| \leq C(d) \|\mu\|_{L^p(\mathbb{R}^d)}$$

$$\rightarrow |\mu(x)| \leq |\mu(\hat{y})| + |\mu(x) - \mu(\hat{y})|$$

$$\leq C(d) \|\mu\|_{L^p(\mathbb{R}^d)} + C(p, d) \|\nabla u\|_{L^p(\mathbb{R}^d)} \cdot 24^{1-\frac{d}{p}}$$

$$\leq C(d, p) \|\mu\|_{1, p, \mathbb{R}^d}$$

Step 3 Assume $\mu \in W^{1,p}(\mathbb{R}^d)$. Take deluxe approximation $\{\mu_n\} \subseteq C_c^\infty(\mathbb{R}^d)$ and observe that in any ball of radius 24 (and actually any radius) they satisfy the assumptions of Ascoli-Arzelà.

The conclusion is standard.

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Remark The previous proof does NOT work if $p = +\infty$ in the point where we apply Hölder inequality.

This case can be dealt with by convolution or simply by observing that Step 1 is almost trivial if $p = +\infty$ (in this case $|\nabla u|$ bounded $\Rightarrow u$ is Lip. with bound on the Lip. const.)

Corollary If $u \in W^{1,p}(\mathbb{R}^d)$ with $p > d$, then

- $\rightarrow u \in C^0(\mathbb{R}^d)$ (and actually uniformly continuous)
- $\rightarrow \lim_{|x| \rightarrow +\infty} u(x) = 0$

↑ This DOES NOT follow from $\int_{\mathbb{R}^d} |u(x)|^p dx < +\infty$

GENERAL FACT If $\int_{\mathbb{R}^d} |v(x)| dx < +\infty$ AND $v(x)$ is unif. cont.

then $v(x) \rightarrow 0$ as $|x| \rightarrow +\infty$

Exercise If $p < d$, then there exists $u \in C^\infty(\mathbb{R}^d) \cap W^{1,p}(\mathbb{R}^d)$ such that $u(x) \rightarrow 0$ as $|x| \rightarrow +\infty$

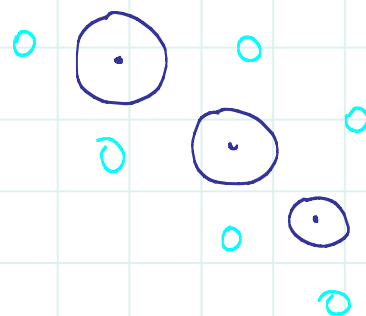
Let us use a scaling technique. Let us consider

$\varphi \in C_c^\infty(B(0,1))$ with $\varphi \geq 0$ and $\int_{B(0,1)} \varphi(x) dx = 1$

Let us consider a sequence of disjoint balls $B(x_m, r_m)$

Let us consider

$$u(x) := a_m \varphi\left(\frac{x-x_m}{r_m}\right) \text{ in } B(x_m, r_m)$$



$$\int_{B(x_m, r_m)} |u(x)|^q dx = a_m^q \int_{B(x_m, r_m)} \varphi\left(\frac{x-x_m}{r_m}\right)^q dx$$

$$= a_m^q r_m^d \int_{B(0,1)} \varphi(y)^q dy$$

$$\nabla u(x) = \frac{a_n}{r_n} \nabla \varphi\left(\frac{x-x_n}{r_n}\right) \text{ and therefore}$$

$$\int_{B(x_n, r_n)} |\nabla u(x)|^q = \frac{a_n^q}{r_n^q} r_n^d \int_{B(0,1)} |\nabla \varphi(y)|^q dy$$

If we want $u \in W^{1,p}(\mathbb{R}^d)$ we need

$$\sum a_n^p r_n^d < +\infty \quad \text{and} \quad \sum a_n^p r_n^{d-p} < +\infty$$

We can take $a_n \equiv 1$ so that $|u(x)| \rightarrow 0$ as $|x| \rightarrow +\infty$
(of course $|x_n| \rightarrow +\infty$)

$$\sum r_n^{\overset{>0}{\textcircled{d}}} < +\infty \quad \text{and} \quad \sum r_n^{\overset{>0}{\textcircled{d-p}}} < +\infty$$

enough to choose $r_n = \frac{1}{n!}$ (can also choose $a_n \rightarrow +\infty$!!)
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