

GAGLIARDO INEQUALITY (1959)Special case of
BRASCAMP-LIEB ineq.Setting: we are in $\mathbb{R}^d$ and we consider d functions

$$\varphi_i \in C_c^\infty(\mathbb{R}^{d-1})$$

Let

$$P_i : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$$

be the projections that "forgets the i -th component".

Define

$$\varphi(x) := \prod_{i=1}^d \varphi_i(P_i x) \quad x \in \mathbb{R}^d$$

For example:

$$d=2 \quad \varphi(x, y) = \varphi_1(y) \varphi_2(x)$$

$$d=3 \quad \varphi(x, y, z) = \varphi_1(y, z) \varphi_2(x, z) \varphi_3(x, y)$$

(General BL ineq. $\varphi(x) := \prod_{i=1}^k \varphi_i(L_i x)$ $L_i : \mathbb{R}^d \rightarrow \mathbb{R}^{m_i}$ linear
with $m_i < d$)**Theorem**

$$\|\varphi\|_{L^1(\mathbb{R}^d)} \leq \prod_{i=1}^d \|\varphi_i\|_{L^{d-1}(\mathbb{R}^{d-1})}$$

ProofInduction on d . We can assume that $\varphi_i \geq 0$. $d=2$

$$\int_{\mathbb{R}^2} \varphi_1(y) \varphi_2(x) dx dy = \left(\int_{\mathbb{R}} \varphi_2(x) dx \right) \cdot \left(\int_{\mathbb{R}} \varphi_1(y) dy \right) \quad \square$$

$$\boxed{3=4} \quad \varphi(x, y, z, w) = a(y, z, w) b(x, z, w) c(x, y, w) d(x, y, z)$$

$$\int_{\mathbb{R}^4} \varphi(x, y, z, w) dx dy dz dw$$

$$= \int_{\mathbb{R}^3} d(x, y, z) \underbrace{\left\{ \int_{\mathbb{R}} a(y, z, w) b(x, z, w) c(x, y, w) dw \right\}}_{\Phi(x, y, z)} dx dy dz$$

$\frac{1}{3}$
 $\frac{2}{3}$

$$\leq \left\{ \int_{\mathbb{R}^3} d(x, y, z)^3 dx dy dz \right\}^{\frac{1}{3}} \left\{ \int_{\mathbb{R}^3} \Phi(x, y, z)^{3/2} dx dy dz \right\}^{2/3}$$

$\parallel d \parallel_{L^3(\mathbb{R}^3)}$

Let us estimate $\Phi(x, y, z)$:

$$\Phi(x, y, z) = \int_{\mathbb{R}} a(y, z, w) b(x, z, w) c(x, y, w) dw$$

$\frac{1}{3}$
 $\frac{1}{3}$
 $\frac{1}{3}$

$$\leq \left\{ \int_{\mathbb{R}} a(y, z, w)^3 dw \right\}^{1/3} \left\{ \int_{\mathbb{R}} b(x, z, w)^3 dw \right\}^{1/3} \left\{ \int_{\mathbb{R}} c(x, y, w)^3 dw \right\}^{1/3}$$

$$\Phi(x, y, z)^{3/2} \leq \underbrace{\left\{ \int_{\mathbb{R}} a(y, z, w)^3 dw \right\}^{1/2}}_{A(y, z)} \underbrace{\left\{ \int_{\mathbb{R}} b(x, z, w)^3 dw \right\}^{1/2}}_{B(x, z)} \underbrace{\left\{ \int_{\mathbb{R}} c(x, y, w)^3 dw \right\}^{1/2}}_{C(x, y)}$$

$$\int_{\mathbb{R}^3} \Phi(x, y, z)^{3/2} dx dy dz \leq \int_{\mathbb{R}^3} A(y, z) B(x, z) C(x, y) dx dy dz$$

$$\leq \left\{ \int_{\mathbb{R}^2} A(y, z)^2 dy dz \right\}^{1/2} \left\{ \int_{\mathbb{R}^2} B(x, z)^2 dx dz \right\}^{1/2} \left\{ \int_{\mathbb{R}^2} C(x, y)^2 dx dy \right\}^{1/2}$$

evolutive hyp

Finally, let us compute any of the three terms

$$\begin{aligned} \int_{\mathbb{R}^2} A(y, z)^2 dy dz &= \int_{\mathbb{R}^2} \left\{ \int_{\mathbb{R}} a(y, z, w)^3 dw \right\} dy dz \\ &= \int_{\mathbb{R}^3} a(y, z, w)^3 dy dz dw \end{aligned}$$

and therefore

$$\left\{ \int_{\mathbb{R}^2} A(y, z)^2 dy dz \right\}^{1/2} = \|a\|_{L^3(\mathbb{R}^3)}^{3/2} \leftarrow \text{cancelled by the } 2/3 \text{ in the integral of } \Phi$$

$d \Rightarrow d+1$ We follow the same idea

Notation $x \in \mathbb{R}^{d+1}$, $\hat{x}_i = x$ without i -th comp.

$\hat{x}_{i, d+1} = x$ without comp. $\#i$ and $\#d+1$

$$\int_{\mathbb{R}^{d+1}} \prod_{i=1}^{d+1} \varphi_i(\hat{x}_i) dx = \int_{\mathbb{R}^d} \varphi_{d+1}(\hat{x}_{d+1}) \underbrace{\left\{ \int_{\mathbb{R}} \prod_{i=1}^d \varphi_i(\hat{x}_i) dx_{d+1} \right\}}_{\substack{\Phi(\hat{x}_{d+1}) \\ \frac{d-1}{d}}} d\hat{x}_{d+1}$$

$$\begin{aligned} &\leq \left\{ \int_{\mathbb{R}^d} \varphi_{d+1}(\hat{x}_{d+1})^d d\hat{x}_{d+1} \right\}^{1/d} \cdot \left\{ \int_{\mathbb{R}^d} \Phi(\hat{x}_{d+1})^{\frac{d}{d-1}} d\hat{x}_{d+1} \right\}^{\frac{d-1}{d}} \\ &= \|\varphi_{d+1}\|_{L^d(\mathbb{R}^d)} \end{aligned}$$

$$\Phi(\hat{x}_{d+1}) = \int_{\mathbb{R}} \varphi_1(\hat{x}_1) \cdots \varphi_d(\hat{x}_d) dx_{d+1}$$

$$\leq \prod_{i=1}^d \left\{ \int_{\mathbb{R}} \varphi_i(\hat{x}_i)^d dx_{d+1} \right\}^{1/d}$$

and therefore

$$\Phi(\hat{x}_{d+1})^{\frac{d}{d-1}} \leq \prod_{i=1}^d \left\{ \int_{\mathbb{R}} \varphi_i(\hat{x}_i)^d dx_{d+1} \right\}^{\frac{1}{d-1}}$$

$$\Phi_i(\hat{x}_{i,d+1})$$

$$\int_{\mathbb{R}^d} \Phi(\hat{x}_{d+1})^{\frac{d}{d-1}} d\hat{x}_{d+1} \leq \int_{\mathbb{R}^d} \underbrace{\prod_{i=1}^d \Phi_i(\hat{x}_{i,d+1})}_{\text{function in } \mathbb{R}^d \text{ which is the product of } d \text{ functions in } \mathbb{R}^{d-1}} d\hat{x}_{d+1}$$

$\leadsto$ we can apply the inductive hyp

$$\leq \prod_{i=1}^d \|\Phi_i(\hat{x}_{i,d+1})\|_{L^{d-1}(\mathbb{R}^{d-1})}$$

Now it is enough to check that

$$\|\Phi_i(\hat{x}_{i,d+1})\|_{L^{d-1}(\mathbb{R}^{d-1})} = \|\varphi_i\|_{L^d(\mathbb{R}^d)}^{\frac{d}{d-1}}$$

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Homothety argument If a pme estimate is true

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C(p,d) \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

then necessarily $q = p^*$

Proof Assume the inequality is true for every $u \in C_c^\infty(\mathbb{R}^d)$.

Then, for every $\lambda > 0$, it is true for $v(x) := u(\lambda x)$

$$\|v\|_{L^q(\mathbb{R}^d)} = \left\{ \int_{\mathbb{R}^d} u(\lambda x)^q dx \right\}^{1/q} = \left\{ \int_{\mathbb{R}^d} u(y)^q \frac{1}{\lambda^d} dy \right\}^{1/q}$$

$y = \lambda x$
 $dy = \lambda^d dx$

$$= \left[\frac{1}{\lambda^{d/q}} \right] \|u\|_{L^q(\mathbb{R}^d)}$$

$\nabla v(x) = \lambda \nabla u(\lambda x)$ and therefore

$$\|\nabla v\|_{L^p(\mathbb{R}^d)}^{1/p} = \left\{ \int_{\mathbb{R}^d} \lambda^p \|\nabla u(\lambda x)\|^p dx \right\}^{1/p}$$

$$= \left[\frac{\lambda}{\lambda^{d/p}} \right] \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

In particular, it has to be true that

$$\frac{1}{\lambda^{d/q}} \|u\|_{L^q(\mathbb{R}^d)} \leq \frac{\lambda}{\lambda^{d/p}} \|\nabla u\|_{L^p(\mathbb{R}^d)} \cdot C(d, p)$$

The two powers of λ should be the same (if not, we get a contradiction when $\lambda \rightarrow 0^+$ or $\lambda \rightarrow +\infty$)

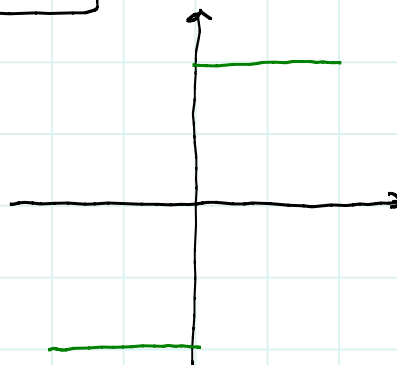
Therefore $\frac{d}{q} = \frac{d}{p} - 1 \leadsto \frac{1}{p} - \frac{1}{q} = \frac{1}{d} \leadsto q = p^*$

In the same way we can show the optimality of the exponents in the pure estimate for the Hölder constant.

The enemy #1 of embeddings and extension

$$\Omega = (-1, 0) \cup (0, 1)$$

$$u \in W^{m,p} \text{ for every } m \geq 1 \text{ and } p$$



But it is not Hölder continuous, and it is not possible to extend it to $\mathbb{R}$ in a Sobolev way.

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