

SOBOLEV EMBEDDINGS

BASIC CASE: WHOLE $\mathbb{R}^d$, $m=1$ Theorem Let us assume that $u \in W^{1,p}(\mathbb{R}^d)$.

$$\frac{1}{p} - \frac{1}{p_*} = \frac{1}{d}$$

Case $p < d$ Then $u \in L^{p_*}(\mathbb{R}^d)$ where
and

$$p_* := \frac{pd}{d-p} \quad (>p)$$

$$\|u\|_{L^{p_*}(\mathbb{R}^d)} \leq c(d,p) \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

PURE ESTIMATE:
only ∇u in RHSAs a consequence, $u \in L^q(\mathbb{R}^d)$ for every $q \in [p, p_*]$ and

$$\|u\|_{L^q(\mathbb{R}^d)} \leq c(d,p,q) \|u\|_{1,p,\mathbb{R}^d}$$

IMPURE ESTIMATE:
full norm in RHSCase $p = d$ Then $u \in L^q(\mathbb{R}^d)$ for every $q \in [p, +\infty)$ and

Excluded

$$\|u\|_{L^q(\mathbb{R}^d)} \leq c(d,p,q) \|u\|_{1,p,\mathbb{R}^d}$$

↑
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equalCase $p > d$ Then $u \in L^\infty(\mathbb{R}^d)$ and

$$\|u\|_{L^\infty(\mathbb{R}^d)} \leq c(d,p) \|u\|_{1,p,\mathbb{R}^d}$$

In addition, u is Hölder continuous and $\in (0,1]$

$$1 - \frac{d}{p}$$

$$|u(y) - u(x)| \leq c(d,p) \|\nabla u\|_{L^p(\mathbb{R}^d)} |y-x|^{1-\frac{d}{p}}$$

PURE ESTIMATE

 $\forall x, y \text{ in } \mathbb{R}^d$

↑ Hölder constant

ITERATED BASIC CASE : WHOLE $\mathbb{R}^d$, ANY m

Theorem Assume that $u \in W^{m,p}(\mathbb{R}^d)$.

$$\frac{1}{p} - \frac{1}{p_*} = \frac{m}{d}$$

Case 1: $mp < d$ In this case $u \in L^{p_*(m)}(\mathbb{R}^d)$ where

and

$$p_*(m) = \frac{dp}{d - pm}$$

$$\|u\|_{L^{p_*(m)}(\mathbb{R}^d)} \leq c(d, p, m) \sum_{|\alpha| = m} \|D^\alpha u\|_{L^p(\mathbb{R}^d)}$$

As a consequence, $u \in L^q(\mathbb{R}^d)$ for all $q \in [p, p_*(m)]$ and

$$\|u\|_{L^q(\mathbb{R}^d)} \leq c(d, p, q, m) \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p(\mathbb{R}^d)}$$

$$\|u\|_{m,p,\mathbb{R}^d}$$

Case 2: $mp = d$ In this case $u \in L^q(\mathbb{R}^d)$ for every $q \in [p, +\infty)$ with improved estimate

Case 3: $mp > d$ First of all, $u \in L^\infty(\mathbb{R}^d)$ with improved estimate. Moreover, let R be the smallest integer such that $Rp > d$. Then $u \in C^{m-R}(\mathbb{R}^d)$ and derivatives of order $m-R$ are Hölder cont. with Hölder-exponent $1 - \frac{d}{Rp}$ and

$$|D^\beta u(x) - D^\beta u(y)| \leq c(d, m, p) \sum_{|\alpha| = m} \|D^\alpha u\|_{L^p(\mathbb{R}^d)} |x - y|^{1 - \frac{d}{Rp}}$$

for every x and y in $\mathbb{R}^d$.

Proof Induction using previous statement.

MORE GENERAL CASE : $m=1$ and $\Omega \subseteq \mathbb{R}^d$ open set

Theorem let us assume that Ω is smooth enough and let $u \in W^{1,p}(\Omega)$.

Case $p < d$ Then $u \in L^q(\Omega)$ for every $q \in [p, p^*]$ and

$$\|u\|_{L^q(\Omega)} \leq c(d, p, q, \Omega) \|u\|_{1,p,\Omega}$$

always the full norm, even if $q = p^*$

Case $p = d$ $u \in L^q(\Omega)$ for every $q \in [p, +\infty)$ with imprecise est.

Case $p > d$ $u \in L^\infty(\Omega)$ and $1 - \frac{d}{p}$ Hölder continuous, with imprecise estimates.

COMPLETELY GENERAL CASE

Assume again that Ω is smooth enough. Assume $u \in W^{m,p}(\Omega)$.

Then conclusions are the same as in the $\mathbb{R}^d$ case, but without pme estimates.

All constants do depend also on Ω .

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EXTENSION PROBLEM

Def Let $\Omega \subseteq \mathbb{R}^d$ be an open set.

① A (m,p) -extender is a map

$$E_{m,p} : W^{m,p}(\Omega) \rightarrow W^{m,p}(\mathbb{R}^d)$$

such that

- (i) $[E_{m,p}(u)](x) = u(x)$ for almost every $x \in \Omega$
- (ii) $E_{m,p}$ is linear
- (iii) we have a "norm control"

$$\|E_{m,p} u\|_{m,p,\mathbb{R}^d} \leq C(m,p,\Omega) \|u\|_{m,p,\Omega}$$

② A m -extender is a map $E_m : L^1_{loc}(\Omega) \rightarrow L^1_{loc}(\mathbb{R}^d)$ such that the restriction of E_m to $W^{m,p}(\Omega)$ is a (m,p) extender for every $p \in [1, +\infty]$.

③ A universal extender is a map $E_{univ} : L^1_{loc}(\Omega) \rightarrow L^1_{loc}(\mathbb{R}^d)$ which is good for every $m \geq 1$ and every $p \in [1, +\infty]$.

First application

If Ω admits a $(1,p)$ extender, then Sobolev embeddings hold true in Ω .

Proof Let us set $\hat{u} := E_{1,p} u$. Then

Sobolev in $\mathbb{R}^d$

$$\begin{aligned} \|u\|_{L^{p^*}(\Omega)} &\stackrel{u=\hat{u} \text{ in } \Omega}{=} \|\hat{u}\|_{L^{p^*}(\Omega)} \leq \|\hat{u}\|_{L^{p^*}(\mathbb{R}^d)} \leq c(d,p) \|\nabla \hat{u}\|_{L^p(\mathbb{R}^d)} \\ &\leq c(d,p,\Omega) \|u\|_{1,p,\Omega} \end{aligned}$$

↑
property of the extender

Very important point If we want embeddings for $W^{m,p}(\Omega)$ it is enough to have $(1,p)$ -extenders

Second application If Ω admits an (m,p) extender, then the deluxe approx. holds true in $W^{m,p}(\Omega)$

Proof Let $\hat{u} = E_{m,p}(u) \in W^{m,p}(\mathbb{R}^d)$.

Assume that we can approximate $\hat{u}$ in $\mathbb{R}^d$ in a deluxe way

$$\begin{array}{c} D^d \hat{u}_m \longrightarrow D^d \hat{u} \quad \text{in } L^p(\mathbb{R}^d) \text{ for every } |d| \leq m \\ \uparrow \\ \in C_c^\infty(\mathbb{R}^d) \end{array}$$

Then the convergence is true a fortiori in $L^p(\Omega)$.

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