

Prop (Sobolev smooth = Sobolev)

Assume that $u \in W^{m,p}(\Omega)$ and $\psi \in C_c^\infty(\Omega)$.

Then $\psi u \in W^{m,p}(\Omega)$.

Rule There is a "nice" formula for derivatives (general Leibniz rule)

$$D^\alpha [\psi u] = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \psi D^{\alpha-\beta} u$$

$\uparrow$
 $\binom{\alpha_1}{\beta_1} \dots \binom{\alpha_d}{\beta_d}$

Prop (Type A partition of the unity)

Let $A \subseteq \mathbb{R}^d$ be an open set, and let $\{A_n\}$ be a family of open sets such that

- (i) $A_n \subset\subset A$ for every $n \geq 1$
- (ii) it is a covering of A , namely

$$\bigcup_{n \geq 1} A_n = A$$

- (iii) the covering is LOCALLY FINITE, namely for every compact set $K \subseteq A$ the set

$\{n \geq 1 : A_n \cap K \neq \emptyset\}$ is finite

Then there exists $\{\psi_n\}$ such that

- (1) $\psi_n \in C_c^\infty(A_n)$
- (2) $\psi_n(x) \geq 0$ for every $n \geq 1$ and every $x \in A_n$ ($x \in \mathbb{R}^d$)
- (3)

$$\sum_{n=1}^{\infty} \psi_n(x) = 1 \quad \forall x \in A$$

$\uparrow$ for every $x \in A$ this is a finite sum because of (iii)

Theorem ($H = W$ by Meyers - Serrin)

Assume that $u \in W^{m,p}(\Omega)$ with $p < +\infty$

Then there exists $\{u_m\} \subseteq C^\infty(\Omega)$ such that
without C

$$u_m \rightarrow u \text{ in } L^p(\Omega)$$

$$D^\alpha u_m \rightarrow D^\alpha u \text{ in } L^p(\Omega) \text{ for every } |\alpha| \leq m.$$

Proof The result is now trivial if $\text{supp}(u) \subset\subset \Omega$.

In this case it is enough to consider the low cost result with $\text{supp}(u) \subset\subset \Omega' \subset\subset \Omega$.

Let us consider now the general case. The goal is the following

$$\forall \varepsilon > 0 \quad \exists v_\varepsilon \in C^\infty(\Omega) \text{ s.t. } \|u - v_\varepsilon\|_{W^{m,p},\Omega} \leq \varepsilon$$

full norm

In order to construct v_ε , we consider Ω_k as before

$$\Omega_k := \{x \in \Omega : \|x\| < k, \text{dist}(x, \partial\Omega) > \frac{1}{k}\} \quad k \geq 1$$

We set $\Omega_0 := \emptyset$ and

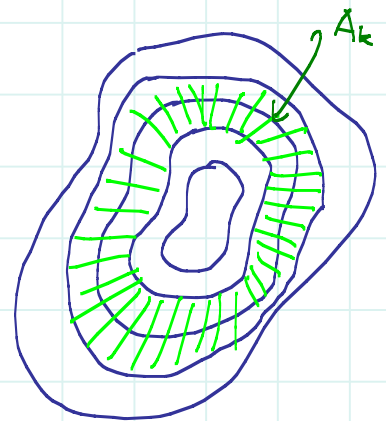
$$A_k := \Omega_{k+1} \setminus \text{clos}(\Omega_k) \quad \forall k \geq 1$$

We observe that $\{A_k\}$ is an open covering of A as in the part. of unity of type A.

(easy check)

Let $\{\psi_k\}$ denote the corresponding part. of the unity, and let us set

$$u_k := \psi_k \cdot u$$



Properties of μ_k .

- $\text{supp}(\mu_k) \subset A_k$
- $\mu_k \in W^{m,p}(\Omega)$ because it is Sobolev-smooth
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$$\sum_{k=1}^{\infty} \mu_k(x) = \mu(x) \quad \text{because } \sum \psi_k(x) = 1$$

↑ this is a finite sum
for every fixed $x \in \Omega$

For every $k \geq 1$, there exists $v_k \in C_c^\infty(A_k)$ such that

$$\|\mu_k - v_k\|_{m,p,\Omega} = \|\mu_k - v_k\|_{m,p,A_k} \leq \frac{\varepsilon}{2^k}$$

(it is enough to choose $v_k = \mu_k * \rho_\delta$ for δ small enough)

Finally define

$$v_\varepsilon(x) := \sum_{k=1}^{\infty} v_k(x) \quad \forall x \in \Omega$$

↑ finite sum for
every fixed $x \in \Omega$

we claim that it works. Indeed

$$\begin{aligned} \|\mu - v_\varepsilon\|_{m,p,\Omega} &= \left\| \sum_{k=1}^{\infty} \mu_k - v_k \right\|_{m,p,\Omega} \\ &\leq \sum_{k=1}^{\infty} \|\mu_k - v_k\|_{m,p,\Omega} \\ &\leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon \end{aligned}$$

Why is $v_\varepsilon \in C^\infty(\Omega)$? Because in every fixed BALL $B(x,r) \subset \Omega$ the sum is FINITE, and therefore $v_\varepsilon \in C^\infty(B(x,r))$ and C^∞ is a local property

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Proof of Smooth Sobolev Induction on m

$m=1$ Claim: $\psi u \in W^{1,p}(\Omega)$ and $D_{x_i}[\psi u] = D_{x_i}\psi \cdot u + \psi \cdot D_{x_i}u$

$$\begin{aligned}
 \int_{\Omega} [\psi u] D_{x_i} \varphi \, dx &= \int_{\Omega} u [\psi D_{x_i} \varphi] \, dx \\
 &\quad \text{test function} \\
 &= \int_{\Omega} u \{ D_{x_i} [\psi \varphi] - D_{x_i} \psi \cdot \varphi \} \, dx \\
 &= \int_{\Omega} u \underbrace{D_{x_i} [\psi \varphi]}_{\text{new test function}} \, dx - \int_{\Omega} u D_{x_i} \psi \varphi \, dx \\
 &= - \int_{\Omega} D_{x_i} u \cdot \psi \varphi \, dx - \int_{\Omega} u D_{x_i} \psi \varphi \, dx \\
 &= - \int_{\Omega} \{ D_{x_i} u \cdot \psi + u \cdot D_{x_i} \psi \} \varphi \, dx \quad \ddot{=}
 \end{aligned}$$

$m \Rightarrow m+1$ Let us consider α with $|\alpha| = m+1$. Let us write $\alpha = \beta + (0, \dots, 0, 1, 0, \dots, 0)$
 $\quad \quad \quad \uparrow$
 $\quad \quad \quad \text{pos. } i$

Now

$$D^{\alpha}[\psi u] = D^{\beta} D_{x_i}[\psi u] = D^{\beta} \left(\underbrace{D_{x_i} \psi}_{\text{smooth}} \cdot \underbrace{u}_{W^{1,p}} + \underbrace{\psi}_{\text{smooth}} \cdot \underbrace{D_{x_i} u}_{W^{1,p}} \right)$$

so the result follows by the inductive hypothesis because $|\beta| = m$.

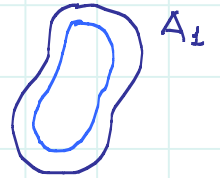
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Lemma Let $\{A_k\}$ be an open covering of A as in the Prop. Then there exists $\{\hat{A}_k\}$ a new covering as in the Prop. with

$$\hat{A}_k \subset\subset A_k \quad \forall k \geq 1$$

"We can take it smaller"

Proof I want to replace A_1 by



$$A_{1,n} := \{x \in A_1 : \text{dist}(x, \partial A_1) > \frac{1}{n}\}$$

Claim: if n is large enough, I can replace A_1 by $\hat{A}_1 := A_{1,n}$. I have to check that

$$A_{1,n} \cup A_2 \cup A_3 \cup \dots = A$$

To this end, it is enough to check that

$$\underbrace{A_{1,n} \cup A_2 \cup A_3 \cup \dots}_{B_n} \supseteq \text{Clos}(A_1) \quad \uparrow \text{clos is not really needed}$$

just observe that B_n is an open covering of $\text{Clos}(A_1)$, which is a compact set. Therefore the conclusion follows because B_n 's are increasing.

In the same way, I can define $\hat{A}_2, \hat{A}_3, \dots$

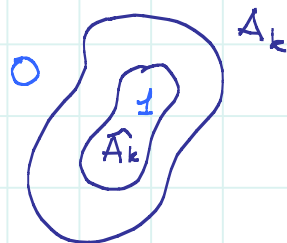
We need to check that $\bigcup_{k \geq 1} \hat{A}_k = A$. Let $x \in A$. Then

x belongs to a finite number of A_k , so if x "survives" after the first k steps, then it survives forever.

Proof of part. of unity

Let us consider $\{\hat{A}_k\}$ as in the Lemma.

For each $k \geq 1$ we can find $\vartheta_k(x)$ s.t.



- $\vartheta_k \in C_c^\infty(A_k)$
- $0 \leq \vartheta_k \leq 1$ in A_k
- $\vartheta_k(x) = 1$ for every $x \in \hat{A}_k$.

Define $S(x) := \sum_{k \geq 1} \vartheta_k(x)$ (> 0 in A because $\{\hat{A}_k\}$ is a covering)

and finally $\psi_k(x) := \frac{\vartheta_k(x)}{S(x)}$

Note that $S(x) \in C^\infty$ because the sum is locally finite

How to find $\vartheta_k(x)$?

Take

$$\hat{A}_k \subset \hat{\hat{A}}_k \subset A_k$$

and then define

$$\vartheta_k = \mathbb{1}_{\hat{\hat{A}}_k} * \rho_\varepsilon \quad \text{with } \varepsilon \text{ small enough}$$

