

MOLLIFIERS AND CONVOLUTION

Def (Mollifier) A mollifier is a function $\rho: \mathbb{R}^d \rightarrow \mathbb{R}$ such that

- (i) some regularity, for example $\rho \in C_c^\infty(\mathbb{R}^d)$
- (ii) $\rho(x) \geq 0$ for every $x \in \mathbb{R}^d$
- (iii) $\rho(x) = 0$ if $\|x\| \geq 1$
- (iv) integral condition

$$\int_{\mathbb{R}^d} \rho(x) dx = 1$$

$\uparrow B(0,1)$

Def (Convolution) Let $\rho: \mathbb{R}^d \rightarrow \mathbb{R}$ be a mollifier
 Let $u \in L^1_{loc}(\mathbb{R}^d)$
 Let $\varepsilon > 0$ be a real number
 For every $x \in \mathbb{R}^d$ we define

$$(u * \rho_\varepsilon)(x) := \int_{\mathbb{R}^d} u(x + \varepsilon y) \rho(y) dy = \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} u(z) \rho\left(\frac{z-x}{\varepsilon}\right) dz$$

$\uparrow B(0,1)$ $\uparrow B(x,\varepsilon)$
 $z = x + \varepsilon y$
 $y = \frac{z-x}{\varepsilon}$
 $dy = \frac{1}{\varepsilon^d} dz$

If we assume just that $u \in L^1(\Omega)$ with $\Omega \subseteq \mathbb{R}^d$ open, then we can consider

$$\hat{u} * \rho_\varepsilon \leftarrow \text{defined for every } x \in \mathbb{R}^d$$

where

$$\hat{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

Prop Properties of the convolution

Let us assume that ρ, u, ε are as in the def. of convolution.

Then the following statements are true

(1) $u * \rho_\varepsilon$ is of class C^∞ (if $\rho \in C^k$, then $u * \rho_\varepsilon \in C^k$ and with some extra work C^{k+1})

(2) $u \rightarrow u * \rho_\varepsilon$ is linear

(3) Assume that $\text{support}(u) \subseteq \Omega$. Then

$$\text{supp}(u * \rho_\varepsilon) \subseteq \text{Clos} \left(\bigcup_{x \in \Omega} B(x, \varepsilon) \right)$$

(4) If $u \in L^p(\mathbb{R}^d)$, then $u * \rho_\varepsilon \in L^p(\mathbb{R}^d)$ and

$$\|u * \rho_\varepsilon\|_{L^p(\mathbb{R}^d)} \leq \|u\|_{L^p(\mathbb{R}^d)}$$

This is true for every $p \in [1, +\infty]$ $\leftarrow$ included

(5) If $u \in L^p(\mathbb{R}^d)$ for some $p \in [1, +\infty)$ $\leftarrow$ excluded then

$$u * \rho_\varepsilon \rightarrow u \quad \text{in } L^p(\mathbb{R}^d)$$

Remark No hope that (5) is true with $p = +\infty$

(consider $\mathbb{1}_{(a,b)}$ in $\mathbb{R}$).

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Exercise Prove the properties.

Proposition (The convolution commutes with w-weak derivatives)

Let $B \subset\subset A \subseteq \mathbb{R}^d$ be open sets.

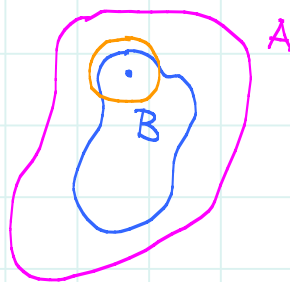
Assume $u \in L^1(A)$ and $D^\alpha u = v \in L^1(A)$

$\uparrow$ w-weak derivative

Assume that

$$0 < \varepsilon < \underbrace{\text{dist}(B, \partial A)}_{\text{No cond. if } A = \mathbb{R}^d}$$

and therefore $B(x, \varepsilon) \subset\subset A$ for every $x \in B$.



Then

$$D^\alpha [\hat{u} * \rho_\varepsilon](x) = (\hat{v} * \rho_\varepsilon)(x) \quad \forall x \in B$$

Proof $(\hat{u} * \rho_\varepsilon)(x) = \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} \hat{u}(z) \rho\left(\frac{z-x}{\varepsilon}\right) dz$

$$D^\alpha (\hat{u} * \rho_\varepsilon)(x) = \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} \hat{u}(z) D_x^\alpha \left[\rho\left(\frac{z-x}{\varepsilon}\right) \right] dz$$

$\uparrow B(x, \varepsilon)$

All $\hat{u}(z)$ are
actually
 $u(z)$

chain rule $\nearrow$ $= \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} \hat{u}(z) [D^\alpha \rho]\left(\frac{z-x}{\varepsilon}\right) \frac{(-1)^{|\alpha|}}{\varepsilon^{|\alpha|}} dz$

$\uparrow B(x, \varepsilon)$

chain rule $\nearrow$ $= \frac{(-1)^{|\alpha|}}{\varepsilon^d} \int_{\mathbb{R}^d} \hat{u}(z) D_z^\alpha \left[\rho\left(\frac{z-x}{\varepsilon}\right) \right] dz$

$\uparrow B(x, \varepsilon) \subseteq A$ class $C_c^\infty(A)$
 $\neq 0$ only in $B(x, \varepsilon) \subset\subset A$

$v = D^\alpha u$ in A $\nearrow$ $= \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} v(z) \rho\left(\frac{z-x}{\varepsilon}\right) dz$

$\uparrow B(x, \varepsilon) \subseteq A$ and $v = \hat{v}$ in A

$$= (\hat{v} * \rho_\varepsilon)(x) \quad \text{Provided that } x \in B$$

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Low-cost approx thm Assume $u \in W^{m,p}(\Omega)$. ($p < +\infty$)

Then there exists $\{u_n\} \subseteq C_c^\infty(\mathbb{R}^d)$ s.t.

$$u_n \rightarrow u \quad \text{in } L^p(\Omega)$$

$$D^\alpha u_n \rightarrow D^\alpha u \quad \text{in } L^p(\Omega') \quad \text{for every } \Omega' \subset\subset \Omega$$

and every $1 \leq |\alpha| \leq m$

Proof Let us consider $u_\varepsilon := \hat{u} * \rho_\varepsilon$

Then $u_\varepsilon \in C^\infty(\mathbb{R}^d)$ (also $C_c^\infty(\mathbb{R}^d)$ if Ω is bounded)

Moreover

$$u_\varepsilon \rightarrow \hat{u} \quad \text{in } L^p(\mathbb{R}^d)$$

and therefore

$$u_\varepsilon \rightarrow \hat{u} = u \quad \text{in } L^p(\Omega)$$

Finally

$$(\hat{D}^\alpha u) * \rho_\varepsilon \rightarrow \hat{D}^\alpha u \quad \text{in } L^p(\mathbb{R}^d)$$

and therefore

$$(D^\alpha u) * \rho_\varepsilon \rightarrow D^\alpha u \quad \text{in } L^p(\Omega')$$

" $\leftarrow$ true only in Ω' and if

$$D^\alpha(u * \rho_\varepsilon) \quad \varepsilon \text{ is small enough}$$

It remains to modify u_ε in order to have the compact support.

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Theorem ($W \Rightarrow H$ for weak derivatives)

Let $u \in L^1_{loc}(\Omega)$ and let $v := D^\alpha u \in L^1_{loc}(\Omega)$.

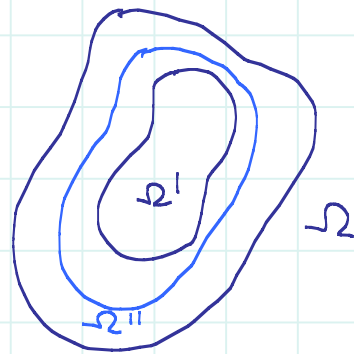
Then there exists $\{u_n\} \subseteq C_c^\infty(\Omega)$ such that

$$u_n \rightarrow u \quad \text{in } L^1_{loc}(\Omega) \quad (\alpha \text{ is fixed})$$

$$D^\alpha u_n \rightarrow v \quad \text{in } L^1_{loc}(\Omega)$$

Proof Let us fix $\Omega' \subset \subset \Omega$ and let us satisfy Ω'

Let us choose $\Omega' \subset \subset \Omega'' \subset \subset \Omega$
(prove that it DOES exist)



Define $u_\varepsilon := \hat{u} * \rho_\varepsilon$
 $\uparrow$
extension to 0
outside Ω''

As in the previous proof we can show that

$$u_\varepsilon \rightarrow u \text{ in } L^p(\Omega'') \text{ and also in } L^p(\Omega')$$
$$D^\alpha u_\varepsilon \rightarrow D^\alpha u \text{ in } L^p(\Omega')$$

How to satisfy any Ω' . For every $k \geq 1$ consider

$$\Omega_k := \{x \in \Omega : \|x\| < k \text{ and } \text{dist}(x, \partial\Omega) > \frac{1}{k}\}$$

It is easy to see that $\bigcup_{k \geq 1} \Omega_k = \Omega$ and for every $\Omega' \subset \subset \Omega$

it turns out that $\Omega' \subset \subset \Omega_k$ eventually.

For every $k \geq 1$ we know how to satisfy Ω_k , in particular there exists v_k s.t.

$$\|u - v_k\|_{L^p(\Omega_k)} \leq \frac{1}{k}$$
$$\|D^\alpha u - D^\alpha v_k\|_{L^p(\Omega_k)} \leq \frac{1}{k}$$

$\nearrow$ it is enough that
 $\searrow$ it goes to 0

I claim that $\{v_k\}$ satisfies any Ω' . Indeed eventually it is true that

$$\|u - v_k\|_{L^p(\Omega')} \leq \|u - v_k\|_{L^p(\Omega_k)} \leq \frac{1}{k} \text{ and the same for derivatives}$$

$\Omega' \subset \subset \Omega_k$
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