

ALGEBRAIC THEOREMS

- Product
- composition 1 (Sobolev inside)
- Absolute value, max/min, truncation
- Composition 2 (Sobolev outside)

Theorem (Product) Let us assume that

$$u \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \quad v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$$

Then $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$D_{x_i}(uv) = D_{x_i} u \cdot v + u \cdot D_{x_i} v$$

Remark we need to assume u and v in L^∞ because otherwise it is not clear that $uv \in L^p$

(in dim 1 it is automatic that $W^{1,p} \Rightarrow L^\infty$, at least on intervals)

General strategy

- Take $u_m \rightarrow u$ and $v_m \rightarrow v$.
- Observe that the formula is true for every $m \in \mathbb{N}$
- Pass to the limit

For every $m \geq 1$ it is true that

$$\int_{\Omega} u_m v_m D_{x_i} \varphi \, dx = - \int_{\Omega} (u_m D_{x_i} v_m + D_{x_i} u_m \cdot v) \varphi \, dx$$

We observe that u_n and v_n can be taken to be equi- L^∞ .

We observe that it is enough to assume that derivatives converge in $L^p(\Omega')$ for a suitable $\Omega' \subset \subset \Omega$ that contains the support of φ .

Consequence: the low cost approx. result is enough.

In addition, L^p conv. $\Rightarrow$ a.e. convergence up to subsequences with L^p domination.

At this point we can pass to the limit in the integrals (pointwise conv. + domination).

This proves the result if $p < +\infty$.

Case $p = +\infty$ In this case we know that

$$u_n \rightarrow u \quad \text{in } L^{2021}(\Omega)$$

$$v_n \rightarrow v$$

$$D_{x_i} u_n \rightarrow D_{x_i} u \quad \text{in } L^{2021}(\Omega')$$

$$D_{x_i} v_n \rightarrow D_{x_i} v$$

and therefore

$$u_n D_{x_i} v_n + v_n D_{x_i} u_n \rightarrow u D_{x_i} v + v D_{x_i} u \quad \text{in } L^{2021}(\Omega')$$

Moreover we know from the approx. theorem that

$u_n, v_n, D_{x_i} u_n, D_{x_i} v_n$ are equi-bounded in $L^\infty(\Omega)$

This implies that $D_{x_i}(uv)$ is in L^∞ .

General fact If $w_n \rightarrow w$ in $L^{2021}(\Omega)$ and

$$\|w_n\|_{L^\infty(\Omega)} \leq M$$

$$\text{then } \|w\|_{L^\infty(\Omega)} \leq M.$$

(Because L^{2021} convergence is pointwise conv. up to subseq.)

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Theorem (Composition with Sobolev inside)

Let us assume that $u \in W^{1,p}(\Omega)$. Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be such that

(i) φ is of class C^1

(ii) $\varphi(0) = 0$ (not needed if Ω is bounded)

(iii) $|\varphi'(x)| \leq L$ for every $x \in \mathbb{R}$.

Then $\varphi(u) \in W^{1,p}(\Omega)$ and

$$D_{x_i} [\varphi(u)] = \varphi'(u(x)) D_{x_i} u(x)$$

Proof Let us approximate $u_n \rightarrow u$ in the low-cost way.

For every $n \geq 1$ and every $\psi \in C_c^\infty(\Omega)$

$$\int_{\Omega} \varphi(u_n(x)) D_{x_i} \psi(x) dx = - \int_{\Omega} \varphi'(u_n(x)) D_{x_i} u_n(x) \psi(x) dx$$

Up to subsequences, the conv. is pointwise almost everywhere.
We just need dominations.

To this end, we observe that

$$\begin{array}{ccc} |\varphi'(u_n(x))| \cdot |D_{x_i} u_n(x)| & & \\ \leq L & \uparrow & L^p \text{ domination} \Rightarrow L^1 \text{ domination} \end{array}$$

As for the LHS, we observe that

$$|\varphi(u_n(x))| = |\varphi(u_n(x)) - \varphi(0)| \leq L |u_n(x)|$$

$\uparrow$ φ is L -lip. cont. $\uparrow$ L^p domination

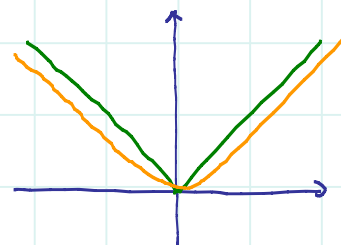
This is ok if $p < +\infty$. Otherwise we argue as before
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Theorem (Absolute value) Let us assume that $u \in W^{1,p}(\Omega)$
Then $|u| \in W^{1,p}(\Omega)$ and

$$D_{x_i} |u| = \text{sign}(u(x)) \cdot D_{x_i} u(x)$$

Proof We approximate the absolute value by means of

$$\varphi_n(\sigma) = \sqrt{\sigma^2 + \frac{1}{n^2}} - \frac{1}{n}$$



We observe that $\varphi_n(\sigma) \rightarrow |\sigma|$ unif. in $\mathbb{R}$
and

$$\varphi_n'(\sigma) \rightarrow \begin{cases} +1 & \text{if } \sigma > 0 \\ -1 & \text{if } \sigma < 0 \end{cases}$$

Consider $u_n(x) := \varphi_n(u(x))$. Due to the previous then
(observe that $\varphi_n(0) = 0$ and $|\varphi_n'(\sigma)| \leq 1$) we know that
 $u_n \in W^{1,p}(\Omega)$ and

$$u_n'(x) = \varphi_n'(u(x)) D_{x_i} u(x)$$

We claim that $u_n(x) \rightarrow |u(x)|$ and $u_n'(x) \rightarrow \text{sign}(u(x)) D_{x_i} u(x)$
in $L^p(\Omega)$.

This follows from the properties of φ_n and φ_n' .
Both cases: dominated pointwise convergence.

Rule $\max\{a, b\} = \frac{a+b+|a-b|}{2}$ and analogously for min.

Therefore: if u and v are in $W^{1,p}$, then $\max\{u, v\}$ and
 $\min\{u, v\}$ are in $W^{1,p}$ and derivatives are what we expect.

$\leadsto$ truncation arguments.

Theorem (Composition - Sobolev outside)

Let A and B be open sets in $\mathbb{R}^d$. Let $u \in W^{1,p}(B)$.
Let

$$\Phi : A \rightarrow B$$

be a function such that

(i) Φ is of class C^1

(ii) Φ is invertible

(iii) J_Φ and $J_{\Phi^{-1}}$ are bounded (in A and B , respectively).

Consider the function

$$v(x) := u(\Phi(x)) \quad x \in A$$

Then $v \in W^{1,p}(A)$ and

$$\frac{\partial v}{\partial x_i}(x) = \sum_{j=1}^d \frac{\partial u}{\partial y_j}(\Phi(x)) \cdot \frac{\partial \Phi_j}{\partial x_i}(x)$$

or even better

$$\nabla v(x) = \nabla u(\Phi(x)) \cdot J_\Phi(x)$$

(Assumption (i) and (ii) and (iii) provide the needed
dominations).

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