

WEAK DERIVATIVES

- Setting:
- $d \geq 1$ (dimension)
 - $\Omega \subseteq \mathbb{R}^d$ open set (any open set)
 - $m \geq 1$ (order of derivation)
 - $p \in [1, +\infty]$ (endpoints included)

Multi-index: $\alpha \in \mathbb{N}^d$ $\alpha = (\alpha_1, \dots, \alpha_d)$
 $|\alpha| := \alpha_1 + \dots + \alpha_d$

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$$

W-weak derivatives

Given $u \in L^1_{loc}(\Omega)$ and $v \in L^1_{loc}(\Omega)$

We say that v is $D^\alpha u$ in the W-sense if

$$\int_{\Omega} u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_{\Omega} v(x) \varphi(x) dx$$

for every test function $\varphi \in C_c^\infty(\Omega)$.

Remark

- (1) u and $v \in L^1_{loc}(\Omega) \Rightarrow$ the integrals are well-defined
- (2) $v(x)$ plays the role of $D^\alpha u(x)$ in the integration by parts formula.

Easy properties

(1) Uniqueness $D^\alpha u$, if it exists, is unique

$$\int_{\Omega} v_1 \varphi dx = (-1)^\alpha \int_{\Omega} u D^\alpha \varphi dx = \int_{\Omega} v_2 \varphi dx$$

$$\Rightarrow \int_{\Omega} (v_1 - v_2) \varphi dx = 0 \quad \forall \varphi \in C_c^\infty; \text{FLCV} \Rightarrow v_1 = v_2 \text{ a.m. ev.}$$

(2) Linearity $D^\alpha (u_1 + u_2) = D^\alpha u_1 + D^\alpha u_2$

(3) Classical compatibility If $u \in C^{|\alpha|}(\Omega)$ then $D^\alpha u$ is what we expect

Less easy properties

(4) "Restriction" If $B \subseteq A$ are open sets, and $v = D^\alpha u$ in A in w-sense, then $v|_B$ is the w- $D^\alpha u|_B$

(5) "Locality". Assume that $u \in L^1(A \cup B)$.

Assume that $v_A = D^\alpha u$ in A

$v_B = D^\alpha u$ in B

$v_A = v_B$ in $A \cap B$

Then u admits $D^\alpha u$ in $A \cup B$ and it is what we expect.

(6) Assume that $u \in L^1_{\text{loc}}(\Omega)$ and $\frac{\partial u}{\partial x_i} = 0$ in w-sense for every $i = 1, \dots, d$ (all 1-st order derivatives)

Then $u(x)$ is locally constant in Ω .

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Stability when passing to limit

Assume $\{u_m\}$ and $\{v_m\}$ are seq. in $L^1_{loc}(\Omega)$.

Assume that

(i) $v_m = D^\alpha u_m$ for every $m \geq 1$

(ii) $u_m \rightarrow u_\infty$ in $L^1_{loc}(\Omega)$

(iii) $v_m \rightarrow v_\infty$ in $L^1_{loc}(\Omega)$

} It is enough that $u_m \rightarrow u_\infty$ and $v_m \rightarrow v_\infty$ weakly in $L^1(\Omega')$ for every $\Omega' \subset\subset \Omega$

Then

$$v_\infty = D^\alpha u_\infty$$

Proof

$$\int_{\Omega} u_m(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_{\Omega} v_m(x) \varphi(x) dx$$

$\downarrow$ $\downarrow$

$$\int_{\Omega} u_\infty(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_{\Omega} v_\infty(x) \varphi(x) dx$$

(It is enough that $\Omega' \supseteq \text{support of } \varphi$)

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H-weak derivatives Again v and u in $L^1_{loc}(\Omega)$.

We say that v is $D^\alpha u$ in H-sense if there exists a sequence

$$\{u_m\} \subseteq C_c^\infty(\Omega)$$

(or $C^\infty(\Omega)$ or $C^{|\alpha|}(\Omega)$)

such that

(i) $u_m \rightarrow u$ in $L^1_{loc}(\Omega)$

(ii) $D^\alpha u_m \rightarrow v$ in $L^1_{loc}(\Omega)$

↑ true derivatives

Easy proposition $H \Rightarrow W$

If $v = D^\alpha u$ in H-sense, then $v = D^\alpha u$ in W-sense.

Theorem (Approximation 0) ($W \Rightarrow H$ for weak derivatives)
If $v = D^\alpha u$ in W -sense, then $v = D^\alpha u$ in H -sense,
namely there exists $\{u_n\} \subseteq C_c^\infty(\Omega)$ such that ...

Sobolev spaces, def. W

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \quad \forall |\alpha| \leq m\}$$

↑
W-weak derivatives

Sobolev spaces, def. H

$$H^{m,p}(\Omega) := \text{exactly the same, with } H\text{-weak derivatives}$$

There is another different possibility

$$\|u\|_{\Omega, m, p} := \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p(\Omega)} \quad \text{FULL NORM}$$

↑
also $(0, \dots, 0)$ is included

$$C^{m,p}(\Omega) := \{u \in C^m(\Omega) : \|u\|_{\Omega, m, p} < +\infty\}$$

It is not difficult to see that $\|u\|_{\Omega, m, p}$ is actually a norm on $C^{m,p}(\Omega)$.

At this point we can define $H^{m,p}(\Omega)$ as the abstract completion of $C^{m,p}(\Omega)$.

Easy theorem $H^{m,p}(\Omega) \subseteq W^{m,p}(\Omega)$

Difficult theorem (Approx. results for Sobolev functions)

Assume that Ω satisfies...

Assume that $u \in W^{1,p}(\Omega)$ with $p < +\infty$

Then there exists $\{u_n\} \subseteq \dots$

such that

$$u_n \rightarrow u \text{ in } L^p(\Omega)$$

and

$$D^\alpha u_n \rightarrow D^\alpha u \text{ in some sense}$$

for every $1 \leq |\alpha| \leq m$.

LOW COST	MEYERS-SERRIN $H=W$ (1964)	DE LUXE
any Ω $u_n \in C_c^\infty(\mathbb{R}^d)$ $D^\alpha u_n \rightarrow D^\alpha u$ in $L^p(\Omega')$ for every $\Omega' \subset\subset \Omega$	any Ω $u_n \in C^\infty(\Omega)$ $D^\alpha u_n \rightarrow D^\alpha u$ in $L^p(\Omega)$	$\partial\Omega$ "regular" $u_n \in C_c^\infty(\mathbb{R}^d)$ $D^\alpha u_n \rightarrow D^\alpha u$ in $L^p(\Omega)$

" u_n is good"
conv. is "bad"
Proof is easy

u_n is rather good
conv. is good
Proof is long

u_n is good
conv. is good
 Ω is NOT any
open set.

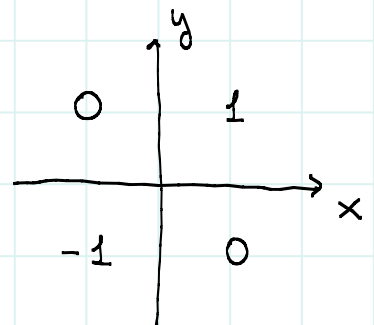
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Example Consider $u(x,y)$ with 3 values as in the figure.

Then $\rightarrow u_{xy} = 0$ in W-sense

$\rightarrow u_x$ and u_y do NOT exist.

(at least in W sense)



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