

Example 1 Discuss existence / uniq. / reg. of functions $u: \mathbb{R} \rightarrow \mathbb{R}$ that are periodic (no prescribed period) such that

$$\ddot{u} = u^5 + \cos x$$

Road map ① If a T -periodic solution exists, then $T = 2k\pi$ for some integer $k \geq 1$.

$$\cos x = \ddot{u}(x) - u^5(x) = T\text{-periodic function} \\ \Rightarrow T = 2k\pi$$

② There exists a solution 2π -periodic

③ There are no other solutions.

Proof of ② $\min \left\{ \int_0^{2\pi} \frac{1}{2} \dot{u}^2 + \frac{1}{6} u^6 + \cos x \cdot u \, dx : u(0) = u(2\pi) \right\}$

Weak Formulation in $H^1((0, 2\pi))$

Compactness

$$L(x, s, p) \geq \frac{1}{2} p^2 + \frac{1}{7} s^6 - A$$

$$M \geq F(u_n) \Rightarrow \int_0^{2\pi} \dot{u}_n(x)^2 \, dx \leq M'$$

$$\Rightarrow \int_0^{2\pi} u_n(x)^6 \, dx \leq M'' \Rightarrow |u_n(x_m)| \leq M'''$$

$$\rightsquigarrow \begin{array}{l} u_{n_k} \rightarrow u_\infty \text{ unif.} \\ \dot{u}_{n_k} \rightarrow \dot{u}_\infty \text{ weakly} \end{array}$$

LSC : as usual (BCs pass to the limit)

Regularity: using $u \in C_c^\infty((0, 2\pi))$ we obtain that any minimizer $u_0 \in H^1$ satisfies

$$(u_0)' = u_0^5 + \cos x$$

$$\leadsto u_0(x) \in C^\infty.$$

Then using perturbations v such that $v(0) = v(2\pi) \neq 0$ we obtain the additional BC

$$u_0(0) = u_0(2\pi) [= 0]$$

In particular, any minimizer solves

↑ Questo NO è
[Aggiunto dopo video]

$$\begin{cases} u_0''(x) = u_0(x)^5 + \cos x \\ u_0(0) = u_0(2\pi) \\ u_0'(0) = u_0'(2\pi) \end{cases}$$

Rmk Up to now, everything work also with 4π instead of 2π .

Let us extend u_0 to $\mathbb{R}$ by 2π -periodicity.

The extension is of class C^1 , but also of class C^2 (here we need that 2π is 2π), and also of class C^3 , and so on

↑ use the equation
in order to compute u_0'' and on

Again, since 2π is a period of $\cos x$, the extension solves the equation for every $x \in \mathbb{R}$.

- ③ Uniqueness In the interval $[0, 2\pi]$ the solution is unique because of the usual argument
- any solution is a minimizer
 - the minimizer is unique due to strict convexity of the Lagrangian.

Assume now that $u_1(x)$ is another solution with period 100π . Then $u_1(x)$ solves the equ. in $[0, 100\pi]$ and therefore it is the unique minimizer in $[0, 100\pi]$.

On the other hand, in $[0, 100\pi]$ we know a solution to the equ, which is the old $u_0(x)$.

By uniqueness in $[0, 100\pi]$ they need to coincide.

Example 2 $\min \left\{ \int_0^1 \dot{u}(x)^6 - u(x)^4 + \arctan u(x) : u(0) = u(1) = 0 \right\}$

The min exists and it is negative.

Take any $u_0(x) \leq 0$ in the interval. consider $u_\varepsilon(x) = \varepsilon u_0(x)$ with $\varepsilon > 0$ small. Then $F(u_\varepsilon) \sim \varepsilon \cdot \text{negative constant}$.

More formally: If $\min = 0$, then $u_0(x) \equiv 0$ is a minimizer, but it is NOT a solution to ELE

$$(6\dot{u}(x)^5)' = -4u(x)^3 + \frac{1}{1+u(x)^2}$$

Existence follows from the direct method. The key point is that

$$L(x, s, p) \geq A|p|^\alpha - B|s|^\beta - C$$

with $A > 0$ and $\beta < \alpha$. Indeed

$M \geq F(u_m)$ we want to obtain $\|u_m\|_{L^\alpha}$. How to obtain?

Due to DBC we observe that

$$|u_m(x)| \leq C_1 + C_2 \cdot \|\dot{u}\|_{L^\alpha}$$

$$|u_n(x)| \leq |u_n(a)| + \left| \int_a^x \dot{u}_n(t) dt \right|$$

$$\leq c_1$$

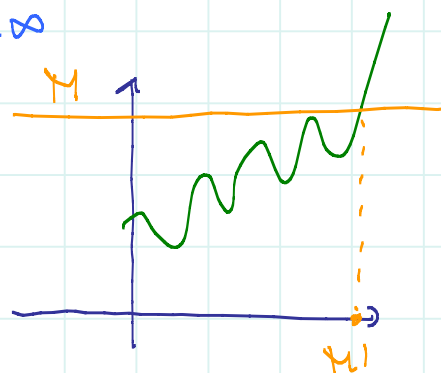
$$|x-a|^{\frac{1}{\alpha}} \cdot \|\dot{u}_n\|_{L^\alpha}$$

$$\frac{1}{\alpha} + \frac{1}{\alpha} = 1$$

At this point

$$M \geq F(u_n) \geq A \int_a^b |\dot{u}_n(x)|^\alpha dx - B(b-a) \{c_1 + c_2 \|\dot{u}_n\|_{L^\alpha}\}^\beta - C(b-a)$$

function of $\|\dot{u}_n\|_{L^\alpha}$ that tends to $+\infty$ as $\|\dot{u}_n\|_{L^\alpha} \rightarrow +\infty$



Example 3 $\int_0^\pi \{ \dot{u}^2 - c u^2 + \arctan u \} dx$

minimize with $u(0) = u(\pi) = 0$.

We know that $\int_0^\pi u(x)^2 dx \leq 1 \cdot \int_0^\pi \dot{u}(x)^2 dx$ if $u(0) = u(\pi) = 0$

If $c < 1$, then we have existence of a minimizer

If $c > 1$, then $\inf = -\infty$

Proof of second claim Consider $u_n(x) = n \sin x$

$$F(u_n) = n^2 \underbrace{\int_0^\pi \cos^2 x dx}_{\frac{\pi}{2}} - c n^2 \underbrace{\int_0^\pi \sin^2 x dx}_{\frac{\pi}{2}} + \int_0^\pi \arctan(\dots)$$

$$= n^2 (1-c) \frac{\pi}{2} + \text{useless guy}$$

Proof if $c < 1$

$$(1-c) \int_0^1 \dot{u}^2 dx + \underbrace{c \int_0^1 (\dot{u}^2 - u^2) dx}_{\geq 0} + \int_0^1 \arctan(u) dx$$

$$M \geq F(u_n) \Rightarrow M \geq (1-c) \int_0^1 (\dot{u}_n^2 + \arctan(u_n)) dx$$

Question: what about $c=1$

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