

## Lagrange Multipliers

Classical setting: open set  $\Omega \subseteq \mathbb{R}^d$

$f: \Omega \rightarrow \mathbb{R}$  and  $g: \Omega \rightarrow \mathbb{R}$  functions of class  $C^1$   
 $x_0 \in \Omega$

$$Z_g := \{x \in \Omega : g(x) = 0\}$$

Theorem Suppose that  $x_0 \in Z_g$  is a min. point of  $f(x)$  restricted to  $Z_g$ , namely

$$f(x_0) \leq f(x) \quad \forall x \in Z_g.$$

Then at least one of the following two cond. is satisfied:

(1)  $\nabla g(x_0) = 0$

(2)  $\exists \lambda \in \mathbb{R}$  such that  $\nabla f(x_0) = \lambda \nabla g(x_0)$

↑ Lagrange multiplier

Several proofs:  $\rightarrow$  explicit function thm

$\rightarrow$  inverse function thm

$\rightarrow$  penalization technique

} lectures of Analisi 2

Theorem Let  $\mathbb{X}$  be an affine space with corresponding vect. sp.  $V$

Let  $F: \mathbb{X} \rightarrow \mathbb{R}$  and  $G: \mathbb{X} \rightarrow \mathbb{R}$  be two functions.

Let  $Z_G := \{x \in \mathbb{X} : G(x) = 0\}$ . Let  $x_0 \in \mathbb{X}$ .

Let us assume that for every  $v \in V$  and  $w \in V$ , the functions

$$\mathbb{R}^2 \ni (t, s) \rightarrow F(x_0 + tv + sw) \in \mathbb{R}$$

$$\mathbb{R}^2 \ni (t, s) \rightarrow G(x_0 + tv + sw) \in \mathbb{R}$$

are of class  $C^1$  (enough in a neigh. of  $(0, 0)$ )

Assume that  $x_0 \in Z_G$  is a min point for the restriction of  $F(x)$  to  $Z_G$ .

Then at least one of the following is true:

(1)  $\delta G(x_0, w) = 0$  for every  $w \in V$

(2)  $\exists \lambda \in \mathbb{R}$  s.t.  $\delta F(x_0, v) = \lambda \delta G(x_0, v) \quad \forall v \in V$ .  
↑  
does NOT depend on  $v$ .

Proof (Based on the inverse function theorem).

Let us consider

$$f(t, s) := F(x_0 + tv + sw)$$

$$g(t, s) := G(x_0 + tv + sw)$$

We observe that

$$g_t(0, 0) = \delta G(x_0, v)$$

$$g_s(0, 0) = \delta G(x_0, w)$$

and analogously for  $f_t(0, 0)$  and  $f_s(0, 0)$ .

Consider the function

$$\mathbb{R}^2 \ni \Phi(t, s) := (f(t, s) - f(0, 0), g(t, s)) \in \mathbb{R}^2$$

$$J\Phi(0, 0) = \begin{pmatrix} \delta F(x_0, v) & \delta F(x_0, w) \\ \delta G(x_0, v) & \delta G(x_0, w) \end{pmatrix}$$

Assume that (1) is false. Then  $\exists w \in V$  s.t.  $\delta G(x_0, w) \neq 0$ .

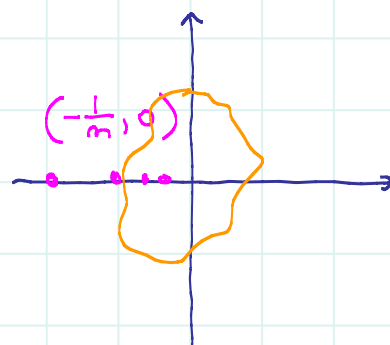
Assume that (2) is false. Then  $J\Phi(0, 0)$  has rank = 2.

Note also that  $\Phi(0, 0) = (0, 0)$ .

By the inverse function theorem the image of  $\Phi$  contains a neighborhood of the origin, in

particular  $(-\frac{1}{n}, 0)$  are in the image of  $\Phi$  when  $n$  is large enough.

This is a pbw. because if  $\Psi$  denotes the inverse of  $\Phi$ , then



$\Psi(-\frac{1}{n}, 0)$  corresponds to a point that tends to  $x_0$  belongs to  $Z_G$ , and has  
 $F(x) = F(x_0) - \frac{1}{n}$ .  
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**THE EXAMPLE** (Poincaré ineq. in an interval).

$$\min \left\{ \underbrace{\int_0^l u(x)^2 dx}_{F(u)} : u(0) = u(l) = 0, \underbrace{\int_0^l u(x)^2 dx = 1}_{G(u)} \right\}$$

**Existence via direct method** weak formulation in  $H^1$

Compactness  $M \geq F(u_n) \Rightarrow \|u_n\|_{L^2(0,l)} \leq M'$

+ DBC  $\Rightarrow \|u_n\|_{L^\infty(0,l)} \leq M''$

$\Rightarrow \exists u_{n_k} \rightarrow u_\infty$  unif. in  $[0, l]$   
 $u_{n_k} \rightarrow u_\infty$  weakly in  $L^2(0, l)$

Both DBCs and integral constraint pass to the limit due to unif. conv.

LSC : trivial because of LSC of the norm.

Regularity Due to Lagrange multipliers we have two possibilities

(1)  $\delta G(u_0, v) = 0 \quad \forall v \in C_c^\infty(0, l)$   
 $\uparrow$   
 minimizer

$$0 = \delta G(u_0, v) = 2 \int_0^l u_0(x) v(x) dx$$

FLCV  $\Rightarrow u_0(x) \equiv 0 \Rightarrow$  DOES NOT SATISFY  $\int_0^l u_0^2 = 1$

$$(2) \delta F(u_0, v) = \lambda \delta G(u_0, v)$$

$$2 \int_0^l \ddot{u}_0(x) \dot{v}(x) dx = 2\lambda \int_0^l u_0(x) v(x) dx$$

In the usual way we obtain

$$(\ddot{u}_0(x))' = -\lambda u_0(x)$$

and we discover that  $u_0 \in C^\infty$ . At this point ELE becomes

$$\left\{ \begin{array}{l} \ddot{u}_0 = -\lambda u_0 \\ u_0(0) = 0 \\ u_0(l) = 0 \\ \int_0^l u_0(x)^2 dx = 1 \end{array} \right\} \quad \text{3 conditions and 3 parameters}$$

Let us determine the sign of  $\lambda$ . Multiply by  $u_0$  and integrate

$$-\lambda \underbrace{\int_0^l u_0(x)^2 dx}_1 = \underbrace{\int_0^l \ddot{u}_0(x) u_0(x) dx}_{\substack{\uparrow \\ \text{boundary terms} = 0}} = - \int_0^l \dot{u}_0(x)^2 dx$$

$\Rightarrow \lambda$  is positive.

The general solution is  $u_0(x) = a \sin(\sqrt{\lambda} x) + b \cos(\sqrt{\lambda} x)$

$$u_0(0) = 0 \Rightarrow b = 0$$

$$u_0(l) = 0 \Rightarrow \underbrace{a}_{\neq 0} \sin(\sqrt{\lambda} l) = 0 \Rightarrow \sqrt{\lambda} l = k\pi \quad k \geq 1$$

$$\Rightarrow \lambda = \frac{k^2 \pi^2}{l^2}$$

The only possibilities are with

$$u_0(x) = a_k \sin\left(\frac{k\pi}{l} x\right) \quad \text{where } a_k \text{ is s.t. } \int_0^l u_0(x)^2 dx = 1$$

$$1 = \int_0^l u_0(x)^2 dx = a_k^2 \int_0^l \sin^2\left(\frac{k\pi}{l}x\right) dx$$

$$x = \frac{ly}{k\pi}$$

$$dx = \frac{l}{k\pi} dy$$

$$= a_k^2 \int_0^{k\pi} \sin^2(y) \frac{l}{k\pi} dy$$

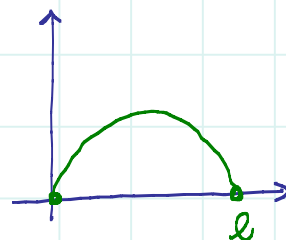
$$= \frac{l}{k\pi} a_k^2 \underbrace{\int_0^{k\pi} \sin^2(y) dy}_{\frac{k\pi}{2}} = \frac{l}{2} a_k^2$$

Conclusion

$$u_0(x) = \sqrt{\frac{2}{l}} \sin\left(\frac{k\pi}{l}x\right)$$

$$u_1(x) = \sqrt{\frac{2}{l}} \frac{k\pi}{l} \cos\left(\frac{k\pi}{l}x\right)$$

Conclusion: the minimum is for  $k=1$



The minimum pbm. is equivalent to an inequality of the form

$$\int_0^l u(x)^2 dx \leq P_l \int_0^l u'(x)^2 dx$$

$$\forall u \in H^1_0((0,l))$$

with  $u(0) = u(l) = 0$

$$\frac{1}{P_l} = \min \text{ that we studied before}$$

Note: the previous computation shows that  $P_l \sim l^2$

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