

ORTHOGONAL OF A SUBSPACE

Theorem Let H be a Hilbert space, and let $V \subseteq H$ be a closed vector subspace.

(1) The projection $P_V : H \rightarrow V$ is well-defined

(2) P_V is linear

(3) For every $x_0 \in H$ the projection $y_0 \in V$ is the unique element of V such that

$$\langle x_0 - y_0, v \rangle = 0 \quad \forall v \in V$$

(4) The ker of P_V is a closed subspace of H equal to

$$V^\perp = \{ w \in H : \langle w, v \rangle = 0 \quad \forall v \in V \}$$

and

$$H = V \oplus V^\perp$$

↑ orthogonal complement

Proof (1) is trivial (just observe that V is convex)

(3) From the general theory we know that y_0 is characterized by

$$\langle x_0 - y_0, y - y_0 \rangle \leq 0 \quad \forall y \in V$$

Given $v \in V$, apply the ineq. with $y = \underset{\in V}{v + y_0}$ and $y = \underset{\in V}{-v + y_0}$ we obtain

$$\langle x_0 - y_0, v \rangle \leq 0 \quad \text{and} \quad \langle x_0 - y_0, -v \rangle \leq 0$$

(2) Linearity Let us show that $P_V(x_1 + x_2) = P_V x_1 + P_V x_2$
"yo"

We know that y_0 is the unique point in V such that

$$\langle x_1 + x_2 - y_0, v \rangle = 0 \quad \forall v \in V$$

On the other hand, we know that $y_0 := P_V x_1 + P_V x_2$ has this property because

$$\langle x_1 + x_2 - P_V x_1 - P_V x_2, v \rangle = \underbrace{\langle x_1 - P_V x_1, v \rangle}_0 + \underbrace{\langle x_2 - P_V x_2, v \rangle}_0 = 0.$$

In the same way we can prove that $P_V(\lambda x_0) = \lambda P_V(x_0)$.

(4) Let $V^\perp$ denote the ker of P_V . It is a vector subspace because P_V is linear, and it is closed because P_V is continuous.

We want to show that

$$V^\perp = \{ w \in H : \underbrace{\langle w, v \rangle}_0 = 0 \quad \forall v \in V \}$$
$$\langle w - P_V w, v \rangle = 0$$

It remains to show that $H = V \oplus V^\perp$.

Clearly $V \cap V^\perp = \{ w \in H : \langle w, w \rangle = 0 \} = \{0\}$.

Given any $x \in H$, we can write

$$x = \underbrace{P_V x}_{\in V} + \underbrace{(x - P_V x)}_{\in V^\perp}$$

why $x - P_V x \in V^\perp$? Again because of (3)

— 0 — 0 —

CONVEX + STRONG $\Rightarrow$ WEAK

Theorem 1 Let $K \subseteq H$. Let us assume that

(1) K is convex

(2) K is strongly closed (if $x_n \rightarrow x_\infty$, and $x_n \in K$ for every n , then $x_\infty \in K$).

Then K is weakly closed (if $x_n \rightharpoonup x_\infty$, and $x_n \in K$ for every n , then $x_\infty \in K$).

we can also allow $+\infty$

Theorem 2 Let $F: H \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function. Let us assume that

(1) F is convex

(2) F is strongly LSC (if $x_n \rightarrow x_\infty$ then $\liminf F(x_n) \geq F(x_\infty)$)

Then F is weakly LSC (if $x_n \rightharpoonup x_\infty$, then the same)

Strong separation POINT 2 CONVEX

Let $K \subseteq H$ be a closed convex set. Let $x_0 \notin K$.

Then there exist $u \in H$ and $\delta_1 < \delta_2$ such that

$$\langle u, y \rangle \leq \delta_1 < \delta_2 = \langle u, x_0 \rangle \quad \forall y \in K$$

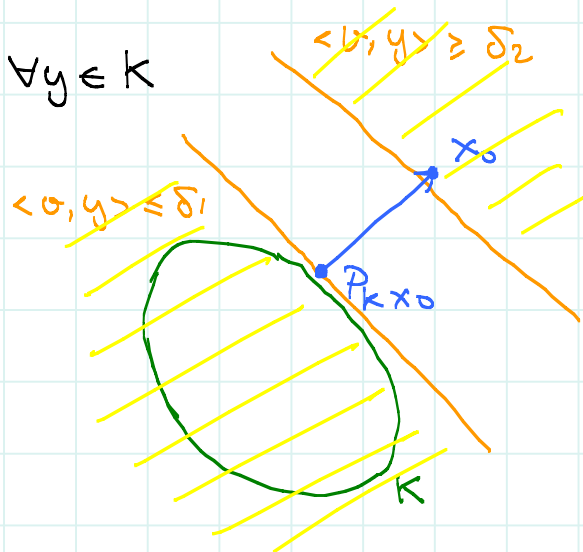
Proof Define

$$u := x_0 - P_K x_0$$

$$\delta_2 := \langle x_0 - P_K x_0, x_0 \rangle$$

$$\delta_1 := \langle x_0 - P_K x_0, P_K x_0 \rangle$$

$$\delta_2 - \delta_1 = \|x_0 - P_K x_0\|^2 > 0$$



It remains to prove that $\langle u, y \rangle \leq \delta_1 \quad \forall y \in K$

namely

$$\langle x_0 - P_K x_0, y \rangle \leq \langle x_0 - P_K x_0, P_K x_0 \rangle$$

namely

$$\langle x_0 - P_K x_0, y - P_K x_0 \rangle \leq 0 \quad \forall y \in K$$

This is the characterization of the projection.
— o — o —

Proof of Theorem 1

Let us consider a sequence $x_n \rightarrow x_\infty$
with $x_n \in K$ for every $n \in \mathbb{N}$.

Claim: $x_\infty \in K$.

Assume this is not the case, namely $x_\infty \notin K$.

Consider a strong separation, namely $v \in H$ s.t.

$$\langle y, v \rangle \leq \delta_2 < \delta_1 = \langle x_\infty, v \rangle \quad \forall y \in K$$

Apply with $y = x_n$:

$$\langle x_n, v \rangle \leq \delta_2 < \delta_1 = \langle x_\infty, v \rangle$$

↓

$$\langle x_\infty, v \rangle$$

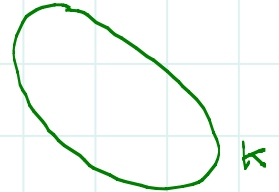
Remark A weak separation would not be enough

$$\langle y, v \rangle < \langle x_\infty, v \rangle \quad \forall y \in K$$

$$\langle x_n, v \rangle < \langle x_\infty, v \rangle$$

STRICT INEQUALITIES DO NOT PASS TO THE LIMIT.

• x_∞



Proof of Theorem 2 let us consider any $x_n \rightarrow x_\infty$.

We need to show that

$$l = \liminf_{n \rightarrow \infty} F(x_n) \geq F(x_\infty).$$

Up to subsequences, we can assume that

$$l = \lim_{n \rightarrow \infty} F(x_n)$$

Consider the sublevels of F :

$$K_\varepsilon := \{x \in H : F(x) \leq l + \varepsilon\}$$

These sets are strongly closed because F is strongly LSC. They are also convex because F is convex.

By theorem 1 they are weakly closed. Since $x_n \in K_\varepsilon$ for every n large enough, we deduce that $x_\infty \in K_\varepsilon$.

This proves that

$$F(x_\infty) \leq l + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, the conclusion follows.

Note: what about the case $l = -\infty$. It can be treated in the same way by considering all sublevels K_M with $M \in \mathbb{R}$

— o — o —

Standard application

Consider

$$F(u) = \int_a^b \varphi(v(x)) dx$$

$u \in L^2$
→ strongly LSC (Fatou's Lemma)
→ convex

Assume that φ is convex with some growth assumption at $\pm \infty$.

Then F is weakly LSC.

— o — o —