

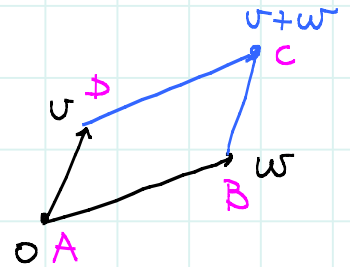
Parallelogram identity] Let H be a Hilbert (it's enough a vector space with a scalar product)

Then

$$\|u+w\|^2 + \|u-w\|^2 = 2\|u\|^2 + 2\|w\|^2 \quad \forall (u, w) \in H^2$$

Proof: trivial

Geom. interpretation



$$AC^2 + BD^2 = 2AD^2 + 2AB^2$$

(sum of $\square$ of diagonals = sum of $\square$ of sides)

Prop. (Jordan - Fréchet - von Neumann) Let V a vector space with a norm that satisfies the parallelogram identity.

Then the norm originates from a scalar product.

Proof] Let us consider

$$p(x, y) := \frac{\|x+y\|^2 - \|x-y\|^2}{4}$$

$$\left[\frac{\|x+y\|^2 - \|x\|^2 - \|y\|^2}{2} \right]$$

Claim: this is a scalar product

Step 0] Trivial facts:

$$\|x\|^2 = p(x, x)$$

$$p(x, 0) = 0$$

$$p(x, y) = p(y, x)$$

$$p(x, -y) = -p(x, y)$$

Step 1 [Remember: we need a factor 2]

$$p(2x, y+z) = 2p(x, y) + 2p(x, z)$$

$$\text{LHS} = \frac{1}{4} \{ \|2x+y+z\|^2 - \|2x-y-z\|^2 \}$$

$$= \frac{1}{4} \{ \|(x+y) + (x+z)\|^2 - \|(x-y) + (x-z)\|^2 \}$$

parall.
identity

$$\begin{aligned} &= \frac{1}{4} \{ 2\|x+y\|^2 + 2\|x+z\|^2 - \cancel{\|y-z\|^2} \\ &\quad - 2\|x-y\|^2 - 2\|x-z\|^2 + \cancel{\|y-z\|^2} \} \end{aligned}$$

$$= \text{RHS} \quad \square$$

Step 2 We prove true additivity

From the previous step with $z=0$ we obtain that

$$p(2x, y) = 2p(x, y)$$

At this point

$$2p(x, y+z) = p(2x, y+z) = 2p(x, y) + 2p(x, z)$$

$\Rightarrow$ additivity

Step 3 It remains to show that $p(\lambda x, y) = \lambda p(x, y)$

- $\lambda \in \mathbb{N} \rightsquigarrow$ additivity + induction
- $\lambda \in \mathbb{Z} \rightsquigarrow \lambda \in \mathbb{N} + \quad p(x, -y) = -p(x, y)$
- $\lambda \in \mathbb{Q} \rightsquigarrow \lambda = \frac{m}{n}$

$$n p\left(\frac{m}{n} x, y\right) = p(mx, y) = n p(x, y) \rightsquigarrow \text{divide by } n$$

- $\lambda \in \mathbb{R}$ This requires some work. We consider two functions

$$\lambda \rightarrow \lambda p(x, y)$$

$$\lambda \rightarrow p(\lambda x, y)$$

They coincide $\forall \lambda \in \mathbb{Q}$. The first one is continuous.
 It remains to show that also the second one is cont.
 It is enough to show that

$\lambda \rightarrow \|\lambda x + y\|$
 is cont. wrt λ , and this is true because

$$\left| \|\lambda_1 x + y\| - \|\lambda_2 x + y\| \right| \leq \|(\lambda_1 - \lambda_2)x\| \leq |\lambda_1 - \lambda_2| \cdot \|x\|$$

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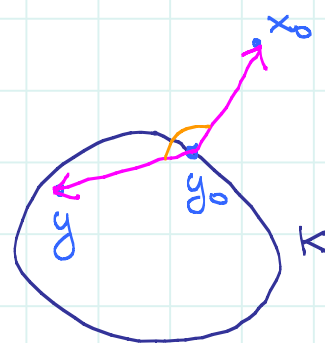
PROJECTION ONTO A CLOSED CONVEX SET

Setting : H is a Hilbert space

$K \subseteq H$ is convex and closed (wrt strong convergence)

For every $x_0 \in H$ we consider

$$\min \{ \|x_0 - y\| : y \in K \}$$



THEOREM Setting as above

(1) The minimum exists for every $x_0 \in H$

(2) The minimizer is a unique point $y_0 \in K$. Let us call it $P_K x_0$

(3) The minimizer y_0 is the unique point $y_0 \in K$ such that

$$\langle x_0 - y_0, y - y_0 \rangle \leq 0 \quad \forall y \in K$$

angle $\in [90^\circ, 180^\circ]$

(4) $\|P_K x_1 - P_K x_2\| \leq \|x_1 - x_2\| \quad \forall (x_1, x_2) \in H^2$
 (1-Lip. continuity)

Proof (1) Let $\{y_n\}$ be an inf-izing sequence, and let

$$I := \inf \{ \|x_0 - y\|^2 : y \in K \}$$

We show it is a Cauchy sequence

$$\begin{aligned} \|y_n - y_m\|^2 &= \|(x_0 - y_n) - (x_0 - y_m)\|^2 \\ &= 2\|x_0 - y_n\|^2 + 2\|x_0 - y_m\|^2 - \|2x_0 - (y_n + y_m)\|^2 \\ &= 2\|x_0 - y_n\|^2 + 2\|x_0 - y_m\|^2 - 4\left\|x_0 - \frac{y_n + y_m}{2}\right\|^2 \end{aligned}$$

$\frac{y_n + y_m}{2} \in K$

Let $\varepsilon > 0$ be fixed. When n and m are large enough, then

$$\|x_0 - y_n\|^2 \leq I + \varepsilon \quad \|x_0 - y_m\|^2 \leq I + \varepsilon$$

$$\left\|x_0 - \frac{y_n + y_m}{2}\right\|^2 \geq I$$

We obtain $\text{LHS} \leq 2I + 2\varepsilon + 2I + 2\varepsilon - 4I = 4\varepsilon$

$\Rightarrow \{y_n\}$ is a Cauchy sequence $\Rightarrow y_n \rightarrow y_\infty \in K$
 $\uparrow$ strongly closed

$\Rightarrow y_\infty$ is a minimizer.

② **Uniqueness** Assume that y_1 and y_2 are two minimizers.

Then $y_1, y_2, y_1, y_2, y_1, y_2, \dots$

is a minimizing sequence

$\Rightarrow$ it is a Cauchy sequence

$\Rightarrow$ it is constant.

Alternative: argue as in Step 1 with $n=1$ and $m=2$.

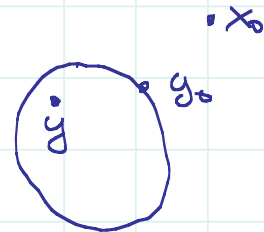
③ Characterization let y_0 be the unique minimizer.

We show that $\langle y - y_0, x_0 - y_0 \rangle \leq 0 \quad \forall y \in K$.

We observe that (remember the place of t)

$$ty + (1-t)y_0$$

is a competitor for every $t \in [0, 1]$, therefore



$$\|x_0 - ty - (1-t)y_0\|^2 \geq \|x_0 - y_0\|^2$$

$$\|(x_0 - y_0) - t(y - y_0)\|^2 \geq \|x_0 - y_0\|^2$$

$$\cancel{\|x_0 - y_0\|^2} + t^2 \|y - y_0\|^2 - 2t \langle x_0 - y_0, y - y_0 \rangle \geq \cancel{\|x_0 - y_0\|^2}$$

Now let $t \rightarrow 0^+$.

We need also the converse. Assume now that

$$\langle x_0 - y_0, y - y_0 \rangle \leq 0 \quad \forall y \in K$$

Claim: y_0 is a minimizer

$$\|x_0 - y_0\|^2 \stackrel{?}{\leq} \|x_0 - y\|^2$$

$$\begin{aligned} \cancel{\|x_0 - y_0\|^2} &\stackrel{?}{\leq} \|x_0 - y_0 + y_0 - y\|^2 \\ &= \cancel{\|x_0 - y_0\|^2} + \|y_0 - y\|^2 - 2 \underbrace{\langle x_0 - y_0, y - y_0 \rangle}_{\geq 0} \end{aligned}$$

④ Lipschitz continuity

$$x_1 \rightsquigarrow P_K x_1$$

$$x_2 \rightsquigarrow P_K x_2$$

$$\langle x_1 - P_K x_1, P_K x_2 - P_K x_1 \rangle \leq 0$$

$$\langle x_2 - P_K x_2, P_K x_1 - P_K x_2 \rangle \leq 0$$

$$\langle P_K x_2 - x_2, P_K x_2 - P_K x_1 \rangle \leq 0$$

↑ the same

Add 1 and 3 and obtain

$$\langle (x_1 - x_2) - (P_k x_1 - P_k x_2), P_k x_2 - P_k x_1 \rangle \leq 0$$

$$- \langle x_1 - x_2, P_k x_1 - P_k x_2 \rangle + \|P_k x_1 - P_k x_2\|^2 \leq 0$$

$$\Rightarrow \|P_k x_1 - P_k x_2\|^2 \leq \langle x_1 - x_2, P_k x_1 - P_k x_2 \rangle$$

$$\leq \|x_1 - x_2\| \cdot \|P_k x_1 - P_k x_2\|$$

$\Rightarrow$ conclusion.

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