

## Weak compactness

## Mysterious Theorem

Let us assume that  $p > 1$ .

Assume that  $\{f_n\} \subseteq L^p(X)$  is BOUNDED.

Then there exists a subsequence that converges weakly to some  $f_\infty \in L^p(X)$

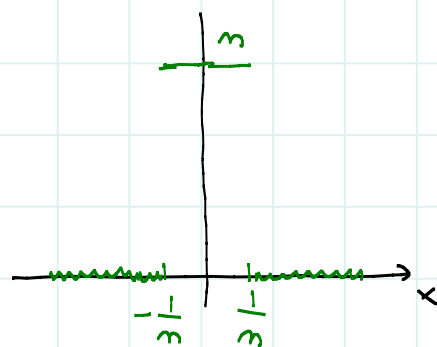
(weakly  $\neq$  if  $p = +\infty$ )

Example The statement is false if  $p = 1$ .

Let us consider  $L^1((-1, 1))$

Let  $\varphi_n(x)$  be as in the figure

$$\varphi_n(x) = n \mathbb{1}_{(-\frac{1}{n}, \frac{1}{n})}(x)$$



It is clear that  $\|\varphi_n\|_{L^1((-1, 1))} = 2$  for every  $n \geq 1$ .

Claim: there exists no converging subsequence in the weak sense.

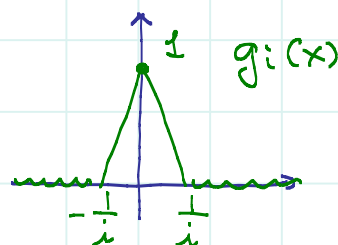
It is easy to see that

$$\lim_{n \rightarrow +\infty} \int_{-1}^1 \varphi_n(x) g(x) dx = 2g(0) \quad \forall g \in C^0([-1, 1])$$

Assume that  $\varphi_{n_k} \rightarrow \varphi_\infty$ . Then this means that

$$\int_{-1}^1 \varphi_\infty(x) g(x) dx = 2g(0) \quad \forall g \in C^0([-1, 1])$$

Consider  $g_i(x)$  as in the figure



$$\int_{-1}^1 \varphi_n(x) g_i(x) dx = 2 g_i(0) = 2$$

$\downarrow$  pointwise conv.  
 $\circ$  dominated by  $|\varphi_n(x)|$ .

More mysterious theorem

Assume that  $\mu(X) < +\infty$ .

Assume that  $\{f_n\} \subseteq L^1(X)$  is BOUNDED

Assume that there exists a continuous function

$$\varphi: \mathbb{R} \rightarrow [0, +\infty)$$

with superlinear growth at  $\infty$ , namely

$$\lim_{p \rightarrow \pm\infty} \frac{\varphi(p)}{|p|} = +\infty$$

such that  $\int_X \varphi(f_n(x)) d\mu$  IS BOUNDED

Then there exists a subsequence that converges weakly in  $L^1(X)$ .

Proof of myst. theorem in the case  $\mu(X) < +\infty$  and  $p \geq 2$

Step 1 Bound in  $L^p(X) \Rightarrow$  Bound in  $L^2(X)$

Step 2  $\exists f_{n_k} \rightharpoonup f_\infty$  weakly in  $L^2(X)$  and also weakly in  $L^1(X)$

Step 3 weak conv. in  $L^1(X)$  + bound in  $L^p(X) \Rightarrow$  weak conv. in  $L^p$ .

Road map to a proof in the general case

Step 1 Consider a countable dense subset  $\{g_i\} \subseteq L^q(X)$

With a diagonal argument we find a subsequence  $\{f_{n_k}\}$  s.t.

$$\lim_{k \rightarrow \infty} \int_X f_{n_k}(x) g_i(x) d\mu \text{ exists for every } i \in \mathbb{N}$$

Step 2 By density and using again the boundedness of  $f_n$  we obtain that

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{X}} f_{n_k}(x) g(x) d\mu = L(g) \quad \forall g \in L^q(\mathbb{X})$$

↑ the limit exists  
and depends linearly on  $g$

Step 3 We would like very much to know that

$$L(g) = \int_{\mathbb{X}} g(x) f_{\infty}(x) d\mu \quad \text{for some } f_{\infty} \in L^p(\mathbb{X}).$$

This requires a REPRESENTATION THEOREM

Example  $F(u) = \int_0^1 (u^4 + \arctan u) dx$

$$\text{with } \{F(u) : u \in C^1([0,1]), u(0) = u(1) = 0\}$$

Weak Formulation

- $u \in W^{1,4}((0,1))$  in this case  $\hat{F}: W^{1,4} \rightarrow \mathbb{R}$
- $u \in H^1((0,1))$  "  $\hat{F}: H^1 \rightarrow \mathbb{R} \cup \{+\infty\}$

Compactness  $M \geq F(u_n) \Rightarrow \|u_n\|_{L^4((0,1))} \leq M + \frac{\pi}{2}$

$$\Rightarrow \exists u_{n_k} \rightharpoonup v_{\infty} \text{ weakly in } L^p((0,1)) \quad \forall p \in [1,4]$$

In the usual way  $\|u_n(x)\|_{L^{\infty}((0,1))} \leq M'$

A.A. At the end of day

$$\begin{aligned} u_{n_k}(x) &\rightarrow u_{\infty}(x) & \forall p \in [1,4] \\ u_{n_k}(x) &\rightarrow u_{\infty}(x) & \text{unif. in } [0,1] \end{aligned}$$

LSC

$$\int_0^1 \arctan(u_n(x)) dx \xrightarrow{\substack{\uparrow \\ \text{unif. conv.}}} \int_0^1 \arctan(u_\infty(x)) dx$$

$$\liminf_{n \rightarrow \infty} \int_0^1 u_n(x)^4 dx \geq \int_0^1 u_\infty(x)^4 dx$$

$\uparrow$   
 $L^1$  weak conv.  
+ convexity of  $p^4$

Regularity  $\varphi(t) = F(u+tv)$  for  $v \in C_c^\infty((0,1))$

$$\varphi'(t) = \int_0^1 \left\{ 4(u+tv)^3 \dot{v} + \frac{1}{1+(u+tv)^2} v \right\} dx$$

To this end, we need a unif. domination. The essential term is

$$(|\dot{u}(x)| + t|\dot{v}(x)|)^3 |\dot{v}(x)|$$

$\uparrow$   $\uparrow$   $\uparrow$   
 $\in L^3$  because  $\text{bounded}$   $\text{bounded}$   
it is in  $L^4$

$$\int_0^1 \left\{ 4\dot{u}^3 \dot{v} + \frac{1}{1+u^2} v \right\} dx = 0 \quad \forall v \in C_c^\infty((0,1))$$

if  $u$  is a minimizer in  $W^{1,4}((0,1))$ . As usual

$$(4\dot{u}_0^3)' = \frac{1}{1+u_0^2} \quad \text{weak form of ELE}$$

$$u_0 \in W^{1,4} \Rightarrow u_0 \in C^0 \Rightarrow \text{RHS} \in C^0 \Rightarrow \dot{u}_0^3 \in C^1$$

$$\Rightarrow \dot{u}_0^3 \in C^0 \Rightarrow \dot{u}_0 \in C^0 \Rightarrow u_0 \in C^1$$

$\uparrow$   
 $\sqrt[3]{\cdot}$  is continuous

we CANNOT say that  $\dot{u}_0 \in C^1$ , because  $\sqrt[3]{C^1} \not\subset C^1$

But on the other hand

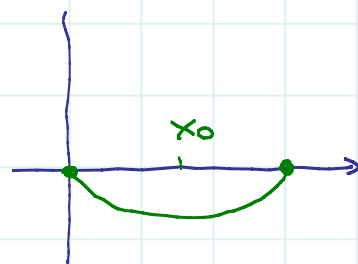
$$\left. \begin{array}{l} u_0 \in C^0 \\ u_0^3 \in \text{Lip} \end{array} \right\} \Rightarrow u_0 \text{ is } \frac{1}{3} - \text{Hölder continuous}$$

and therefore  $u_0 \in C^{1,1/3}$

( $C^1$  with deriv.  $\frac{1}{3}$  Hölder cont.)

In this case the solution is exactly  $C^{1,1/3}$

The solution admits at least one min. point  $x_0$



Claim  $\lim_{x \rightarrow x_0} \frac{u_0(x)^3}{x - x_0} \neq 0$

$$= \lim_{x \rightarrow x_0} \frac{u_0(x)^3 - u_0(x_0)^3}{x - x_0} \underset{\text{HöP}}{=} \lim_{x \rightarrow x_0} \frac{[u_0(x)^3]^1}{1} \underset{\substack{\uparrow \\ \text{use ELE}}}{\neq 0}$$

— 0 — 0 —