

weak convergence in L^p

Setting $L^p(X, m, \mu)$ with $p \in [1, +\infty]$.

Defn Let $\{f_n\} \subseteq L^p(X)$ and let $f_\infty \in L^p$. Let q denote the conjugate exponent $\frac{1}{p} + \frac{1}{q} = 1$.
We say that

$$\begin{aligned} f_n &\rightarrow f_\infty && \text{weakly in } L^p && \text{if } p < +\infty \\ f_n &\xrightarrow{*} f_\infty && \text{weakly } * \text{ in } L^\infty && \text{if } p = +\infty \end{aligned}$$

if

$$\lim_{n \rightarrow +\infty} \int_X f_n(x) g(x) d\mu = \int_X f_\infty(x) g(x) d\mu$$

for every $g \in L^q(X)$.

Remark It is the generalization of the usual case $p=q=2$.

Special case Assume that $\mu(X)$ is finite. For every $1 \leq p_1 < p_2 \leq +\infty$ it turns out that

$$L^{p_2}(X) \subseteq L^{p_1}(X)$$

and as a consequence

$$f_n \rightarrow f_\infty \text{ weakly in } L^{p_2}(X) \Rightarrow f_n \rightarrow f_\infty \text{ weakly in } L^{p_1}(X)$$

(this is because $L^{q_2}(X) \supseteq L^{q_1}(X)$)

Remark If $\mu(X)$ is finite, then weak conv. in $L^1(X)$ is the "weakest" possible.

On the contrary, the strongest weak conv. is weak $*$ convergence in $L^\infty(X)$

Prop Let us assume that $\{f_n\} \subseteq L^p(X)$ is BOUNDED.
Let $W \subseteq L^q(X)$ be a subset such that $\text{Span}(W)$ is dense.
Let us assume that

$$\lim_{n \rightarrow +\infty} \int_X f_n(x) g(x) d\mu = \int_X f_\infty(x) g(x) d\mu \quad \forall g \in W$$

Then $f_n \rightarrow f_\infty$ weakly in L^p (weakly $*$ if $p = +\infty$).

Proof Consider

$$\begin{aligned} \left| \int_X f_n g - \int_X f_\infty g \right| &\leq \left| \int_X f_n g - \int_X f_n g_\varepsilon \right| \\ &\quad + \left| \int_X f_n g_\varepsilon - \int_X f_\infty g_\varepsilon \right| \\ &\quad + \left| \int_X f_\infty g_\varepsilon - \int_X f_\infty g \right| \end{aligned}$$

$$\leq \underbrace{\|f_n\|_{L^p(X)}}_{\leq M} \|g - g_\varepsilon\|_{L^q(X)} + \left| \int_X f_n g_\varepsilon - \int_X f_\infty g_\varepsilon \right| + \|f_\infty\|_{L^p} \cdot \|g - g_\varepsilon\|_{L^q}$$

If $g_\varepsilon \in \text{Span}(W)$ then the second term goes to 0 as $n \rightarrow +\infty$.
Therefore

$$\limsup_{n \rightarrow +\infty} \left| \int_X f_n g - \int_X f_\infty g \right| \leq \{M + \|f_\infty\|_{L^p(X)}\} \|g - g_\varepsilon\|_{L^q(X)}.$$

Since $\text{Span}(W)$ is dense, and g_ε is arbitrary, then the LHS is 0.
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Corollary Assume again that $\mu(X)$ is bounded.

Assume that $\{f_n\} \subseteq L^p(X)$ is bounded and that

$$f_n \rightarrow f_\infty \text{ in } L^1(X) \text{ (the weakest possible)}$$

Then

$$f_n \rightarrow f_\infty \text{ in } L^p(X) \text{ (with } * \text{ if } p = +\infty)$$

Proof $L^\infty(X)$ is dense in $L^1(X)$.

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Theorem (LSC of convex integrands)

Let us assume that $\mu(X) < +\infty$.

Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a convex function that is bounded from below

Let us assume that

$$f_n \rightarrow f_\infty \text{ weakly in } L^1(X)$$

Then

$$\liminf_{n \rightarrow +\infty} \int_X \varphi(f_n(x)) d\mu \geq \int_X \varphi(f_\infty(x)) d\mu$$

Proof Let us consider the special case where

$$\rightarrow \varphi \in C^1(\mathbb{R})$$

$$\rightarrow \varphi' \text{ is bounded}$$

Let us consider the usual inequality

$$\varphi(a+b) \geq \varphi(a) + \varphi'(a)b \quad \forall (a,b) \in \mathbb{R}^2$$

Let us apply it with $a = f_\infty(x)$ and $b = f_n(x) - f_\infty(x)$.

We obtain that

$$\int_X \varphi(f_n(x)) d\mu \geq \int_X \varphi(f_\infty(x)) d\mu + \int_X \underbrace{\varphi'(f_\infty(x))}_{\in L^\infty(X) \text{ because } \varphi' \text{ is bounded}} (f_n(x) - f_\infty(x)) d\mu$$

If we let $n \rightarrow \infty$ we obtain what we need.

There are two ways to approach the general case

→ by the general principle "STRONG + CONVEX $\Rightarrow$ WEAK"

→ by hand with an approximation result.

Lemma Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a convex function.

Then there exists a family $\{\varphi_\varepsilon\}_{\varepsilon>0}$ with the following properties

(i) $\varphi_\varepsilon: \mathbb{R} \rightarrow \mathbb{R}$ is convex for every $\varepsilon>0$,

(ii) $\varphi_\varepsilon \in C^1(\mathbb{R})$ and φ'_ε is bounded for every $\varepsilon>0$ (with a bound that depends on ε)

(iii) $\varphi_\varepsilon(x) \nearrow \varphi(x)$ as $\varepsilon \rightarrow 0^+$, namely for every fixed $x \in \mathbb{R}$ the function $\varepsilon \rightarrow \varphi_\varepsilon(x)$ is nonincreasing w.r.t ε .

Example $\varphi(x) = x^4$

Take $\varphi_\varepsilon(x) = \varphi(x)$ in $[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]$

and then the tangent lines.



Proof of LSC given the Lemma For every $\varepsilon>0$ we observe that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\mathbb{X}} \varphi(f_n(x)) d\mu &\geq \liminf_{n \rightarrow \infty} \int_{\mathbb{X}} \varphi_\varepsilon(f_n(x)) d\mu \\ &\geq \int_{\mathbb{X}} \varphi_\varepsilon(f_\infty(x)) d\mu \end{aligned}$$

We conclude by considering the supremum with respect to $\varepsilon>0$ (this requires monotone convergence thm (Beppo Levi thm), and hence some boundedness from below).

Example 1 Consider $f_n(x) = \cos(nx)$

This converges weakly to 0 in every $L^p((a,b))$

Proof It is enough to prove that

$$f_n \xrightarrow{*} 0 \quad \text{weakly } * \text{ in } L^\infty((a,b))$$

Need to check $\int_a^b f_n(x) g(x) dx \rightarrow 0 \quad \forall g \in L^1((a,b))$

This is almost trivial if $g(x) = \mathbb{1}_{(c,d)}(x)$ with $(c,d) \subseteq (a,b)$
but this class spans a dense subset of L^1 .
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