

VARIATIONAL APPROACH TO BOUNDARY VALUE PROBLEMS

Example

$$\begin{cases} u'' = \arctan u \\ u(0) = 3, u(1) = 6 \end{cases}$$

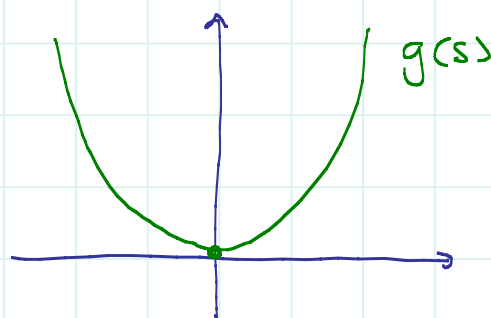
Goal: discuss existence / uniqueness / regularity of solutions
 $u: [0, 1] \rightarrow \mathbb{R}$

Strategy: look for a variational problem for which the given pbm. is ELE and then apply the direct method.

$$(L_p)' = L_s \quad L(x, s, p) = \frac{1}{2} p^2 + g(s) \quad \text{where } g'(s) = \arctan s$$

Consider the min. pbm.

$$\min \left\{ \int_0^1 \left[\frac{1}{2} u'^2 + g(u) \right] dx : \text{DBC} \right\}$$



1. Weak formulation

$$F(u) = \int_0^1 \left[\frac{1}{2} u'^2 + g(u) \right] dx \quad \forall u \in H^1((0,1))$$

$F: H^1((0,1)) \rightarrow \mathbb{R}$ Also DBCs make sense in H^1

2. Compactness

$$M \geq F(u_m) \geq \frac{1}{2} \int_0^1 u_m'^2 dx$$

$$\Rightarrow \|u_m'\|_{L^2}^2 \leq 2M \quad L^2 \text{ bound on derivatives}$$

$\Rightarrow$ unif. estimate on $\frac{1}{2}$ -Hölder constant of u_m

Together with DBC, we obtain

$$|u_n(x)| \leq M' \quad \forall x \in [a, b] \quad L^\infty \text{ bound on functions}$$

In conclusion there exists a subsequence s.t.

$$u_{n_k}(x) \rightarrow u_\infty(x) \quad \text{uniformly} \quad (\text{Ascoli - Arzelà})$$

$$i_{n_k}(x) \rightarrow v_\infty(x) \quad \text{weakly } L^2 \quad (\text{weak cpt. of balls})$$

$$\text{With the usual trick} \quad v_\infty(x) = i_\infty(x)$$

We observe that DBC pass to the limit.

3. LSC If $u_n \rightarrow u_\infty$ unif. then

$$\lim_{n \rightarrow +\infty} \int_0^1 g(u_n(x)) dx = \int_0^1 g(u_\infty(x)) dx$$

If $i_n \rightarrow i_\infty$ weakly L^2 , then

$$\liminf_{n \rightarrow +\infty} \int_0^1 i_n(x)^2 dx \geq \int_0^1 i_\infty(x)^2 dx$$

4. Regularity Let $u_0 \in H^1((0, 1))$ be a minimizer.

For every $v \in C_c^\infty((0, 1))$ and every $t \in \mathbb{R}$ we define

$$\varphi(t) := F(u_0 + tv) = \int_0^1 (i_0 + t\dot{v})^2 + g(u_0 + tv) dx$$

and we hope that

$$\varphi'(t) = \int_0^1 \{ 2(i_0 + t\dot{v})\dot{v} + \arctan(u_0 + tv) \cdot v \} dx$$

Let us check the domination

$$2 (|u_0(x)| + |t| \cdot |\dot{u}(x)|) |\dot{v}(x)| + |\arctan(\dots)| \cdot |v(x)|$$

$\uparrow$ $\uparrow_{t \in [-1,1]}$ $\uparrow$ bounded $\uparrow$ bounded $\uparrow$ bounded
 $\in L^1((0,1))$ because $\in L^2((0,1))$

Since u_0 is a minimizer, then

$$0 = \varphi'(0) = \int_0^2 \{ 2u_0 \dot{v} + \arctan(u_0) \cdot v \} dx$$

which is equivalent to

$$\int_0^2 \underbrace{2u_0}_{f} \underbrace{\dot{v}}_{\varphi} dx = - \int_0^2 \underbrace{\arctan(u_0)}_g \cdot \underbrace{v}_{\varphi} dx$$

$\leadsto g$ is the weak derivative of f , namely

$$\underbrace{(2u_0)'}_{\substack{\uparrow \text{weak deriv. of} \\ \text{weak deriv.}}} = \arctan(u_0)$$

From H^1 to C^2

$$u_0 \in H^1 \Rightarrow u_0 \in C^0 \Rightarrow \text{RHS} \in C^0$$

$$\Rightarrow 2u_0 \in C^1 \Rightarrow u_0 \in C^2$$

$\uparrow$
classical
compatibility

In this moment, $u_0(x)$ is a solution of the equation in the classical sense

Bootstrap $\leadsto u_0 \in C^\infty([0,1])$.

We proved existence and regularity.

UNIQUENESS Two possible approaches

→ use the functional

→ use the equation

VIA FUNCTIONAL This requires two steps

1. Proving that every solution to the equ. is a minimizer

2. " " the minimizer is unique

Both points follow in some way from the convexity of the Lagrangian.

Every competitor u can be written as $u = u_0 + v$, where $v \in H^1((0,1))$ with $v(0) = v(1) = 0$.

$$F(u_0 + v) = \underbrace{\int_0^1 u_0'^2}_{\text{orange}} + 2 \int_0^1 u_0' v + \int_0^1 v'^2 + \int_0^1 g(u_0 + v) dx$$
$$\geq \underbrace{\int_0^1 g(u_0) dx}_{\text{orange}} + \int_0^1 \arctan u_0 \cdot v$$

$g(a+b) \geq g(a) + b g'(a) \nearrow$

$$= F(u_0) + \int_0^1 v'^2 dx + \underbrace{\int_0^1 \{2 u_0' v + \arctan u_0 \cdot v\} dx}_{=0 \text{ if } u_0 \text{ satisfies E-L-E}}$$

$$\geq F(u_0)$$

even in weak sense

(we need int. by parts in H^1)

$\Rightarrow u_0$ is a minimizer

Moreover we have equality $\Leftrightarrow v' \equiv 0 \Leftrightarrow v \equiv \text{constant}$

$$\Leftrightarrow v \equiv 0$$

$\Rightarrow$ the minimizer is unique.

VIA EQUATION Assume we have two solutions

$$v'' = \arctan v$$

$$w'' = \arctan w$$

$$(v-w)'' = \arctan v - \arctan w$$

Multiply by $v-w$ and integrate:

$$\int_0^1 (v-w)(v-w)'' dx = \int_0^1 \underbrace{(v-w)(\arctan v - \arctan w)}_{\geq 0} dx$$

" ≥ 0

$$-\int_0^1 (\dot{v}-\dot{w})^2 dx + \underbrace{[(v-w)(\dot{v}-\dot{w})]}_{\uparrow 0} \Big|_{x=0}^{x=1}$$

$$\Rightarrow \int_0^1 (\dot{v}-\dot{w})^2 dx \leq 0 \Rightarrow \dot{v}-\dot{w} \equiv 0 \Rightarrow \dots \Rightarrow v \equiv w$$

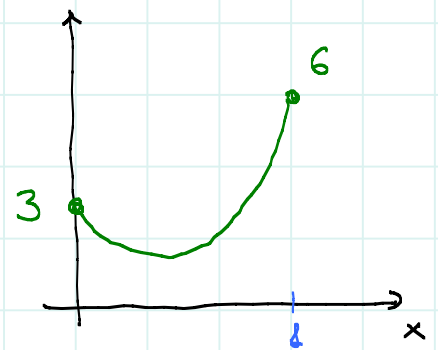
Remark The convexity of $g(s)$ is equivalent to the monotonicity of $\arctan s$.

Qualitative properties of the solution

$$\begin{cases} u'' = \arctan u \\ u(0) = 3, u(1) = 6 \end{cases}$$

The solution is strictly positive and (therefore) convex

$u_0 \geq 0$: truncation argument!!
(we need that g has a min in 0)



Want to exclude that $u_0(x_0) = 0$ at some $x_0 \in (0, 1)$

In this case, x_0 would be a min point, therefore $\dot{u}_0(x_0) = 0$

But in this case u_0 solves

$$\begin{cases} \ddot{u}_0 = \arctan u_0 \\ u_0(x_0) = 0 \\ \dot{u}_0(x_0) = 0 \end{cases}$$

Cauchy problem with unique soln
 $u_0(x) \equiv 0$.

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