

$$(2\dot{u}_0(x))' = -\sin(u_0(x)^3) \cdot 3u_0(x)^2$$

↑
weak derivative
of weak derivative

4.2 From H^1 to C^1 .

$$u_0 \in H^1([a,b]) \Rightarrow u_0 \in C^0([a,b]) \Rightarrow \text{RHS} \in C^0([a,b])$$

$$\Rightarrow 2\dot{u}_0(x) \in C^1([a,b]) \quad (\text{classical compatibility: if } f' \in C^0, \text{ then } f \in C^1)$$

$$\Rightarrow \dot{u}_0(x) \in C^0([a,b]) \Rightarrow u_0(x) \in C^1([a,b])$$

If we exploit that $u_0(x)$ is the antiderivative of $\dot{u}_0(x)$ we obtain also that $u_0(x) \in C^2([a,b])$.

At this point ELE is

$$2\ddot{u}_0(x) = -3u_0(x)^2 \sin(u_0(x)^3)$$

4.3 From C^2 to C^∞ is the usual BOOTSTRAP ARGUMENT

$$u_0 \in C^2 \Rightarrow \text{RHS} \in C^2 \Rightarrow \ddot{u}_0 \in C^2 \Rightarrow u_0 \in C^4 \Rightarrow \text{RHS} \in C^4$$

and so on.

In order to justify the computation of $\varphi'(t)$, we need a domination for the integrand in

$$\int_a^b \{ 2[\dot{u}_0(x) + t \dot{v}(x)] \dot{v}(x) - \sin(u_0(x) + t v(x))^3 \cdot 3(u_0(x) + t v(x))^2 \cdot v(x) \} dx$$

$\uparrow$ in L^1 because $\dot{u}$ is in L^2 $\uparrow$ bounded $\uparrow$ bounded

Example 1 $F(u) = \int_a^b \{ \dot{u}(x)^2 + u(x) \cos u(x) \} dx + \text{DBC}$

Weak formulation in H^1 .

Compactness $M \geq \int_a^b \{ \dot{u}_m(x)^2 + u_m(x) \cos(u_m(x)) \} dx$

$$\geq \int_a^b \{ \dot{u}_m(x)^2 - |u_m(x)| \} dx = (*)$$

As in the previous example

$$|u_m(x)| \leq |u_m(a)| + \int_a^x |\dot{u}_m(t)| dt$$

$$\leq \underbrace{|u_m(a)|}_A + |b-a|^{1/2} \left\{ \int_a^b |\dot{u}_m(t)|^2 dt \right\}^{1/2}$$

In other words $|u_m(x)| \leq A + |b-a|^{1/2} \|\dot{u}_m\|_{L^2(a,b)}$
and therefore

$$M \geq \|\dot{u}_m\|_{L^2(a,b)}^2 - (b-a) [A + |b-a|^{1/2} \|\dot{u}_m\|_{L^2(a,b)}]$$

= second order polynomial in $\|\dot{u}_m\|_{L^2(a,b)}$



From this inequality we deduce that

$$\|u_m\|_{L^2((a,b))} \leq M'$$

From now on we proceed as before.

GENERAL FACT Assume that

$$L(x, s, p) \geq A p^2 - B |s|^{\alpha} \quad \text{with } A > 0 \text{ and } \alpha < 2$$

and that we have at least one DBC. Then

$$F(u_m) \leq M \quad \Rightarrow \quad \|u_m\|_{L^2((a,b))} \leq M'$$



Example 1 bis $\min \left\{ \int_a^b (u'^2 + u \cos u) dx : u \in C^1 \right\}$

Inf $= -\infty$!! u very big and constant with $\cos u = -1$

Example 1 ter $\min \left\{ \int_a^b (u'^2 + u^3) dx : u \in C^1, u(a) = u(b) = 0 \right\}$

Consider $u_0(x) \geq 0$ and satisfies the DBC. Then consider

$$u_n(x) = -n u_0(x)$$

$$F(u_n) = n^2 \int_a^b u_0(x)^2 dx - n^3 \int_a^b u_0(x)^3 dx$$

Example 2 $\min \left\{ \int_a^b (\dot{u}^2 + (u-x)^{20}) dx : u \in C^1 \text{ NO DBC} \right\}$

... Compactness $M \geq F(u_n) \geq \int_a^b \dot{u}_n(x)^2 dx$

$\leadsto \dot{u}_{n_k} \rightarrow v_{\infty}$

Now we need a unif bound on $|u_n(x)|$ in order to apply A.A.
We know that

$$M \geq \int_a^b (u(x)-x)^{20} dx = (b-a) \underbrace{(u(x_n)-x_n)^{20}}_{\substack{\uparrow \\ \text{mean value} \\ \text{thm for integrals}}} \quad \underbrace{\uparrow}_{\substack{\text{mysterious point} \\ \text{in } [a,b]}} u_n(x_n) - x_n$$

$\leadsto (u_n(x_n) - x_n)^{20} \leq \frac{M}{b-a}$

$|u_n(x_n) - x_n| \leq M' \leadsto |u_n(x_n)| \leq |x_n| + M' \leq M''$

At this point, as before

$$\begin{aligned} |u_n(x)| &\leq |u_n(x_n)| + \left| \int_{x_n}^x \dot{u}_n(t) dt \right| \\ &\leq M'' + |b-a|^{1/2} \|\dot{u}_n\|_{L^2(a,b)} \leq M''' \end{aligned}$$

GENERAL FACT Assume that

$$L(x, s, p) \geq Ap^2 + \psi(s) - B$$

this implies that $\psi(s) \geq -C$

where $A > 0$ and $\lim_{s \rightarrow \pm\infty} \psi(s) = +\infty$ without DBCs.

Then $F(u_n) \leq M$ implies $\|\dot{u}_n\|_{L^2} \leq M'$ and $|u_n|_{L^\infty} \leq M''$

Another possibility for LSC is the following result

$$\liminf_{n \rightarrow +\infty} \int_a^b \psi(x, u_n(x)) dx \geq \int_a^b \psi(x, u_\infty(x)) dx$$

is true provided that $u_n(x) \rightarrow u_\infty(x)$ (it is enough in a pointwise sense) and that $\psi(x, s)$ is bounded from below

This is FATOU lemma. (Classical example: $\psi(x, s) = \frac{1}{s^2}$).

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