

ROAD MAP OF THE DIRECT METHOD

Setting: S is a set, and $F: S \rightarrow \mathbb{R}$ is a function

Goal: Proving that

$$\min \{ F(s) : s \in S \}$$

exists.

Road map ① Weak formulation

② Compactness

③ Lower semicontinuity

④ Regularity

Weak formulation We introduce a set $\hat{S} \supseteq S$ and

$$\hat{F}: \hat{S} \rightarrow \mathbb{R} \cup \{+\infty\} \quad \leftarrow \text{let us allow it, it is our friend}$$

such that $\hat{F}(s) = F(s) \quad \forall s \in S$
 $\leq \leftarrow$ this is enough

Compactness We look for a notion of convergence in $\hat{S}$ with the following property:

if $\{s_n\} \subseteq \hat{S}$ and $\hat{F}(s_n) \leq M$, then there exists $s_{n_k} \rightarrow s_\infty$

LSC We hope that $\hat{F}$ is LSC wrt the same notion of conv.
 more precisely, if $s_n \rightarrow s_\infty$ in $\hat{S}$ and $\hat{F}(s_n) \leq M$, then

$$\liminf_{n \rightarrow +\infty} \hat{F}(s_n) \geq \hat{F}(s_\infty)$$

At this point we already know that

$$\min \{ \hat{F}(s) : s \in \hat{\mathcal{S}} \}$$

exists (Weierstrass theorem) [De Giorgi: every pbm. has a solution, if we relax enough the notion of solution].

Regularity Let $s_0 \in \hat{\mathcal{S}}$ be a min. point. We want to prove that $s_0 \in \mathcal{S}$ and $\hat{F}(s_0) = F(s_0)$.
If we can do this, then s_0 is a minimizer for the original pbm

$$F(s_0) = \hat{F}(s_0) \leq \hat{F}(s) \leq F(s) \quad \forall s \in \mathcal{S}$$

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Model case $\mathcal{S} := C^1([a, b])$ with $u(a) = 7$ and $u(b) = 13$

$$F(u) := \int_a^b [u'(x)^2 + \cos(u(x)^3)] dx \quad \forall u \in \mathcal{S}$$

① Weak formulation We consider [Notion: $H^1 = H^{1,2} = W^{1,2}$]

$$\hat{\mathcal{S}} := \{ u \in H^1((a, b)) : u(a) = 7, u(b) = 13 \}$$

$$\hat{F}(u) = \int_a^b [\underset{\substack{\uparrow \\ \text{weak derivative}}}{u'(x)^2} + \cos(u(x)^3)] dx \quad \forall u \in \hat{\mathcal{S}}$$

Remarks:

- boundary values of $u \in H^1$ are well defined
- also $\cos(u(x)^3)$ is integrable because u is cont.
- in this case $\hat{F}$ has real values

2) Compactness let $\{u_n\} \subseteq \mathcal{S}$ be such that $\hat{F}(u_n) \leq M$

$$M \geq \int_a^b [u_n'(x)^2 + \cos(u_n(x)^3)] dx$$

$$\geq \int_a^b [u_n'(x)^2] dx - (b-a)$$

$$\Rightarrow \int_a^b u_n'(x)^2 dx \leq M + (b-a)$$

Weak compactness of balls $\Rightarrow \exists u_{n_k}(x) \xrightarrow{\text{weak convergence in } L^2(a,b)} v_\infty(x)$

We recall that $u_n(x)$ are $\frac{1}{2}$ -Hölder continuous, and in particular

$$|u_n(x) - u_n(a)| \leq \int_a^x |u_n'(t)| dt$$

$$\leq \int_a^b |u_n'(t)| dt$$

$$\leq |b-a|^{1/2} \left\{ \int_a^b |u_n'(t)|^2 dt \right\}^{1/2}$$

$$\leq |b-a|^{1/2} \cdot M'$$

In particular

$$|u_n(x)| \leq |u_n(a)| + |b-a|^{1/2} \cdot M' \quad \forall x \in [a,b]$$

$\uparrow$ Fixed because of DBC $\uparrow$ number

$$\Rightarrow |u_n(x)| \leq M'' \quad \forall x \in [a,b] \text{ and for every } n.$$

So we have

- a uniform bound on $|\mu_n(x)|$
- a uniform bound on the $\frac{1}{2}$ -Hölder constant of μ_n

Ascoli - Arzelà thm $\Rightarrow \{\mu_n(x)\}$ has a subsequence that converges uniformly in $[a, b]$.

Important point: if we apply Ascoli - Arzelà to $\{\mu_{n_k}(x)\}$ that we found before, we obtain a further subsequence $\{\mu_{n_{k_i}}(x)\}$ such that

$$\dot{\mu}_{n_{k_i}}(x) \longrightarrow v_\infty(x) \quad \text{weak in } L^2([a, b])$$

$$\mu_{n_{k_i}}(x) \longrightarrow \mu_\infty(x) \quad \text{uniform in } [a, b]$$

Fortunately $v_\infty(x) = \mu_\infty'(x)$ weak derivative

$$\begin{aligned} \int_a^b \mu_{n_{k_i}}(x) \varphi'(x) dx &= - \int_a^b \dot{\mu}_{n_{k_i}}(x) \varphi(x) dx \\ \downarrow \qquad \qquad \qquad \downarrow \\ \int_a^b \mu_\infty(x) \varphi'(x) dx &= - \int_a^b v_\infty(x) \varphi(x) dx \end{aligned}$$

$\forall \varphi \in C_c^\infty([a, b])$ Note that DBC pass to the limit due to unif. conv.

③ LSC Assume that $\mu_n \rightarrow \mu_\infty$ unif.
 $\dot{\mu}_n \rightarrow \dot{\mu}_\infty$ weakly

Can we conclude that

$$\liminf_{n \rightarrow \infty} \hat{F}(\mu_n) \geq F(\mu_\infty)$$

This follows from

$$\liminf_{n \rightarrow +\infty} \int_a^b u_n(x)^2 dx \geq \int_a^b u_\infty(x)^2 dx \quad \text{LSC of the norm}$$

$$\lim_{n \rightarrow +\infty} \int_a^b \cos(u_n(x)^3) dx = \int_a^b \cos(u_\infty(x)^3) dx \quad \text{unif. conv.}$$

[4] Now we know that a minimizer $u_0(x)$ exists in $H^1(a,b)$

[4.1] weak form of ELE

[4.2] from H^1 to C^1

[4.3] from C^1 to C^∞

[4.1] We consider $\varphi(t) := \hat{F}(u_0 + t \overset{C_c^\infty(a,b)}{v})$

$$= \int_a^b [u_0(x) + t v(x)]^2 + \cos((u_0 + t v)^3)$$

$$\varphi'(t) = \int_a^b \{ 2[u_0(x) + t v(x)] v(x) - \sin(u_0(x) + t v(x))^3 \cdot 3(u_0(x) + t v(x))^2 \cdot v(x) \} dx$$

$$0 = \varphi'(0) = \int_a^b \{ 2 \underset{\substack{\uparrow \\ \text{weak derivative}}}{u_0(x)} v(x) - \sin(u_0(x)^3) 3 u_0(x)^2 v(x) \} dx$$

$$\int_a^b \underbrace{2 u_0(x)}_{f(x)} \underbrace{v(x)}_{\dot{\varphi}(x)} dx = - \int_a^b \underbrace{\{ -\sin(u_0(x)^3) 3 u_0(x)^2 \}}_{g(x)} \underbrace{v(x)}_{\varphi(x)} dx$$

This implies that $2u_0(x)$ admits a weak derivative and

$$\boxed{(2u_0(x))'} = -\sin(u_0(x)^3) \cdot 3 u_0(x)^2$$

$\uparrow$
 weak derivative
 of weak derivative