

Weak derivatives - Definition 1

Defn Let u and v be in $L^1_{loc}((a,b))$. We say that v is the H -weak deriv. of u if there exists a sequence $\{u_n\} \subseteq C^\infty((a,b))$ such that (asking $u_n \in C^1((a,b))$ is the same)
 $\uparrow$ no cpt. supp. requirement
 $u_n \rightarrow u$ and $u'_n \rightarrow v$ in $L^1_{loc}((a,b))$
 $\uparrow$ classical

Exercise Prove uniqueness of the H -weak derivative.

Prop. ($H \Rightarrow W$) The H -deriv. is also a W -deriv.

Proof

$$\int_a^b u_n(x) \varphi'(x) dx = - \int_a^b u'_n(x) \varphi(x) dx$$

$\downarrow$

$$\int_a^b u(x) \varphi'(x) dx = - \int_a^b v(x) \varphi(x) dx \quad \square$$

SOBOLEV SPACES

$p \in [1, +\infty]$ (endpoints included)

Defn $W^{1,p}((a,b)) = \{ f \in L^p((a,b)) : f' \in L^p((a,b)) \}$
 $\uparrow$ W weak deriv. of f

$H^{1,p}((a,b)) = \dots$ the same with $f' = H$ weak deriv.

Remark By the previous result $H^{1,p}((a,b)) \subseteq W^{1,p}((a,b))$

Regularity of $W^{1,p}$ functions

Theorem (Hölder continuity) Assume that $p > 1$ and let q denote the conjugate exponent $\frac{1}{p} + \frac{1}{q} = 1$.

Then any function $u \in W^{1,p}((a,b))$ is $\frac{1}{q}$ -Hölder continuous.
(Lip. cont. if $p = +\infty$).

Proof Let v denote the w.d. of u . Then for every $a \leq x < y \leq b$:

$$|u(y) - u(x)| = \left| \int_x^y v(t) dt \right| = \left| \int_a^b v(t) \underbrace{1_{(x,y)}}_{\frac{1}{p}}(t) dt \right|$$

$$\leq \underbrace{\left\{ \int_a^b |v(t)|^p dt \right\}^{1/p}}_{\text{number}} \cdot |y-x|^{1/q}$$

Hölder inequality

The case $p = +\infty$ is even simpler.

Important remark If $u \in W^{1,p}((a,b))$ (without "loc")

then " u is defined and continuous up to the boundary" and therefore it makes sense to consider pointwise values

$$u(x) \quad \forall x \in [a, b]$$

Remark In the case $p=1$ we have in general no Hölder cont., but continuity up to the boundary, and therefore uniform continuity.

Theorem $W^{1,p}((a,b)) \subseteq H^{1,p}((a,b))$ (and therefore they are equal)

and W -weak deriv. are H -weak derivatives

TRUE ONLY
IF $p < +\infty$

Proof Let v be the W -derivative of u .

Idea: approximate v by a sequence $\{v_m\} \subseteq C^\infty((a,b))$.

Define, for some $x_0 \in (a,b)$,

↑ HERE WE NEED
 $p < +\infty$

$$u_m(x) := u(x_0) + \int_{x_0}^x v_m(t) dt \quad \forall x \in [a,b]$$

↑
pointwise values
are well defined

It is trivial that $u_m \in C^\infty((a,b))$, and that we can choose $v_m \rightarrow v$ in the same space where v lies

We need to prove that

$u_m \rightarrow u$ in the sense we need ($L^1_{loc}((a,b))$ or $L^p((a,b))$).

The idea is that convergence is better

$$|u(x) - u_m(x)| = \left| \int_{x_0}^x (v(t) - v_m(t)) dt \right| \leq \int_{x_0}^x |v(t) - v_m(t)| dt$$

↑
at least if $x \geq x_0$

Depending on the assumptions on v , we can conclude that $u_m(x) \rightarrow u(x)$ (if $v_m \rightarrow v$ in L^p , we need Hölder inequality)

If there is no "loc", the convergence is even uniform.

Remark $H^{1,\infty}((a,b)) \subsetneq W^{1,\infty}((a,b))$

↑
 $C^1((a,b))$ with

u and u' bounded

↑ equal to the set of
Lipschitz functions

(L^∞ limit is uniform limit
and unif. limits preserve continuity)

Integration by parts

Assume that u and v are in $W^{1,1}((a,b))$
(note that $W^{1,p}((a,b)) \subseteq W^{1,1}((a,b))$)

Then

$$\int_a^b u(x) v'(x) dx = u(b)v(b) - u(a)v(a) - \int_a^b u'(x) v(x) dx$$

where u' and v' are the weak derivatives.

Proof Use H-approx.

$$\begin{array}{ll} u_n \xrightarrow{L^1} u & u_n'(x) \xrightarrow{L^1} u'(x) \\ v_n \xrightarrow{L^1} v & v_n'(x) \xrightarrow{L^1} v'(x) \end{array}$$

actually
uniform, and this
is enough

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