

ℓ^2 and L^2 spaces

$$\ell^2 = \{ \text{sequences } \{a_n\}_{n \geq 1} \subseteq \mathbb{R} \text{ such that } \sum_{n=1}^{\infty} a_n^2 < +\infty \}$$

$$\Omega \subseteq \mathbb{R}^d \text{ is measurable, then } L^2(\Omega) = \{ f: \Omega \rightarrow \mathbb{R} \text{ s.t. } \int_{\Omega} f(x)^2 dx < +\infty \}$$

More generally we can define

$$L^2(X, \mathcal{M}, \mu)$$

$\uparrow$ set $\uparrow$ σ -algebra $\nwarrow$ measure ($\mu \geq 0$, $+\infty$ is allowed, countably additive)

$$\text{in } \ell^2 : \langle \{a_n\}, \{b_n\} \rangle := \sum_{n=1}^{\infty} a_n b_n$$

$$\text{in } L^2 : \langle f, g \rangle := \int_{\Omega} f(x) g(x) dx$$

Remark ℓ^2 is equal to $L^2(\mathbb{N})$ with respect to the measure

$$\mu(A) = \text{number of el. of } A \quad \forall A \subseteq \mathbb{N}.$$

If I is an uncountable set, then $L^2(I)$

is a Hilbert space which is NOT separable

$\uparrow$ with the same measure as above

(Exercise: provide a Hilbert basis in this case)

Remark $|f(x)g(x)| \leq \frac{1}{2}|f(x)|^2 + \frac{1}{2}|g(x)|^2$ and this implies that $\langle f, g \rangle$ is well-defined

How to prove completeness of L^2

Let $\{f_n\} \subseteq L^2$ be a Cauchy sequence.

Step 1 Enough to prove existence of conv. subseq.

Step 2 (Acceleration lemma) There exists a subseq (not relabeled) such that

$$\|f_{n+1} - f_n\| \leq \frac{1}{2^n}$$

Step 3 Observe that $f_n = f_1 + (f_2 - f_1) + (f_3 - f_2) + \dots + (f_n - f_{n-1})$ and therefore $\{f_n\}$ converges if and only if

$$\sum_{n=1}^{\infty} (f_{n+1} - f_n) \text{ converges}$$

DO NOT EVEN THINK OF USING THE LAST RESULT OF LECT. 11

Step 4 The functions

$$\hat{S}_n(x) := \sum_{k=1}^n |f_{k+1}(x) - f_k(x)|$$

are monotone w.r.t n and equi-bounded in L^2 (use acceleration)

Step 5 $\{\hat{S}_n(x)\}$ converges in the pointwise sense and the limit is in L^2 and dominated by a L^2 function (Beppo-Levi)

Step 6 Since the series with $|\cdot|$ converges, the series without absolute values converges.

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Proof that l^2 is complete I denote elements of l^2 as functions $f: \mathbb{N} \rightarrow \mathbb{R}$.

Let $\{f_n(i)\}$ be a Cauchy sequence in l^2 .

Claim $f_n(i) \rightarrow f_\infty(i)$ in l^2 .

Step 1 There exists $f_\infty: \mathbb{N} \rightarrow \mathbb{R}$ such that

$$\lim_{n \rightarrow \infty} f_n(i) = f_\infty(i) \quad \forall i \in \mathbb{N}$$

For every $m > n$ and every $i \geq 1$ it turns out that

$$|f_m(i) - f_n(i)|^2 \leq \sum_{k=1}^{\infty} |f_m(k) - f_n(k)|^2 = \|f_m - f_n\|_{l^2}^2$$

$\Rightarrow \{f_n(i)\}$ is a Cauchy sequence of number for every fixed $i \in \mathbb{N}$

$\Rightarrow$ It has a limit which we call $f_\infty(i)$.

Step 2 We prove that $f_\infty \in l^2$.

For every $n \geq 1$:

$$\sum_{i=1}^n |f_\infty(i)|^2 = \sum_{i=1}^n \lim_{n \rightarrow \infty} |f_n(i)|^2$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n |f_n(i)|^2$$

$$\leq \limsup_{n \rightarrow \infty} \|f_n\|_{l^2}^2$$

$$\leq M \quad (\text{general fact: Cauchy sequences are bounded})$$

Step 3 (Uniform smallness of tails)

$$\forall \varepsilon > 0 \exists m \in \mathbb{N} \forall n \in \mathbb{N} \sum_{i=m+1}^{+\infty} |f_n(i)|^2 \leq \varepsilon$$

Since $\{f_n\}$ is a Cauchy sequence, there exists $m_0 \in \mathbb{N}$ s.t.

$$\|f_m - f_n\|_{\ell^2}^2 \leq \frac{1}{4} \varepsilon \quad \forall m \geq m_0 \quad \forall n \geq m_0$$

Let $m \in \mathbb{N}$ be such that

$$\sum_{i=m+1}^{+\infty} |f_m(i)|^2 \leq \frac{1}{4} \varepsilon \quad \forall m \in \underbrace{\{1, \dots, m_0\}}_{\text{only a FINITE number}}$$

Claim: this m is good for every $n \in \mathbb{N}$. Let $n > m_0$, then

$$\begin{aligned} \sum_{i=m+1}^{+\infty} |f_n(i)|^2 &= \sum_{i=m+1}^{+\infty} |f_n(i) - f_{m_0}(i) + f_{m_0}(i)|^2 \\ &\leq 2 \sum_{i=m+1}^{+\infty} |f_n(i) - f_{m_0}(i)|^2 + 2 \underbrace{\sum_{i=m+1}^{+\infty} |f_{m_0}(i)|^2}_{\leq \frac{1}{4} \varepsilon} \\ &\leq 2 \|f_n - f_{m_0}\|_{\ell^2}^2 \\ &\leq 2 \cdot \frac{1}{4} \varepsilon \\ &= \varepsilon. \end{aligned}$$

Step 3.5 The same holds true for f_∞ , namely

$$\sum_{i=m+1}^{+\infty} |f_\infty(i)|^2 \leq \varepsilon$$

↑
same m as before

Same proof of step 2.

Step 4 $f_n \rightarrow f_\infty$ in the ℓ^2 sense

$$\begin{aligned}\|f_n - f_\infty\|^2 &= \sum_{i=1}^{\infty} |f_n(i) - f_\infty(i)|^2 \\ &= \underbrace{\sum_{i=1}^n |f_n(i) - f_\infty(i)|^2}_{\rightarrow 0 \text{ as } n \rightarrow +\infty} + \sum_{i=n+1}^{\infty} |f_n(i) - f_\infty(i)|^2 \\ &\quad \text{because of} \\ &\quad \text{pointwise convergence}\end{aligned}$$

$$\sum_{i=n+1}^{\infty} |f_n(i) - f_\infty(i)|^2 \leq 2 \sum_{i=n+1}^{\infty} |f_n(i)|^2 + 2 \sum_{i=n+1}^{\infty} |f_\infty(i)|^2$$

Usual conclusion

→ Fix $\varepsilon > 0$

→ Choose n is such a way that last 2 terms $\leq \frac{1}{2} \varepsilon$
for every n

→ Choose n_0 such that first term $\leq \frac{1}{2} \varepsilon$ for every $n \geq n_0$.

Example Consider $H = L^2((0, 2\pi))$

Claim: the sequence $\{\sin(nx)\}$ converges weakly to 0.

It is enough to prove that $\int_0^{2\pi} f(x) \sin(nx) dx \rightarrow 0$

$\forall f \in W \leftarrow$ span a dense subset of $L^2((0, 2\pi))$.

$W =$ charact. funct. of intervals !!! $\int_a^b \sin(nx) dx$

$$= -\frac{1}{n} (\cos(nb) - \cos(na)) \rightarrow 0.$$