

Prop. (weak conv. and scalar product)

Let H be a Hilbert space. Let us assume that

(i) $u_n \rightarrow u_\infty$ strongly

(ii) $w_n \rightarrow w_\infty$ weakly

(iii) $\{w_n\}$ is bounded (one day this will follow from (ii))

Then

$$\lim_{n \rightarrow \infty} \langle u_n, w_n \rangle = \langle u_\infty, w_\infty \rangle.$$

Proof $|\langle u_\infty, w_\infty \rangle - \langle u_n, w_n \rangle|$

$\langle u_n, w_\infty - w_n \rangle$
" "

$[\leq |\langle u_\infty, w_\infty \rangle - \langle u_n, w_\infty \rangle| + |\langle u_n, w_\infty \rangle - \langle u_n, w_n \rangle|]$

(Bad choice: try it!!)

$\leq \underbrace{|\langle u_\infty, w_\infty \rangle - \langle u_\infty, w_n \rangle|}_{\substack{\uparrow \\ \text{FIXED}}} + \underbrace{|\langle u_\infty, w_n \rangle - \langle u_n, w_n \rangle|}_{\substack{\leq \|w_n\| \cdot \|u_\infty - u_n\| \\ \uparrow \text{c.s.} \\ \leq M \|u_\infty - u_n\| \\ \downarrow 0}}$

Therefore

$$\lim_{n \rightarrow \infty} |\langle u_\infty, w_\infty \rangle - \langle u_n, w_n \rangle| = 0$$

— 0 — 0 —

THINGS THAT DO NOT WORK IF $\dim H = +\infty$

① $u_n \rightarrow u_\infty, w_n \rightarrow w_\infty \not\Rightarrow \langle u_n, w_n \rangle \rightarrow \langle u_\infty, w_\infty \rangle$

$$e_n \rightarrow 0, e_n \rightarrow 0$$

$$\langle e_n, e_n \rangle = \|e_n\|^2 = 1 \not\rightarrow 0$$

② The norm is NOT cont. wrt weak conv. $u_n \rightarrow u_\infty \not\Rightarrow \|u_n\| \rightarrow \|u_\infty\|$

$$u_n = e_n$$

③ Closed balls are not STRONGLY COMPACT

④ Spheres are NOT WEAKLY CLOSED

↑ $\{v \in H: \|v - v_0\|^2 = R^2\}$ Again $\|e_n\| = 1$ and $e_n \rightarrow 0$.
— 0 — 0 —

Theorem 1 (LSC of the norm) If $v_n \rightarrow v_\infty$ weakly in H , then

$$\liminf_{n \rightarrow \infty} \|v_n\| \geq \|v_\infty\|$$

Proof $\|v_n\|^2 = \|(v_n - v_\infty) + v_\infty\|^2$

$$= \|v_n - v_\infty\|^2 + \|v_\infty\|^2 + 2 \langle v_n - v_\infty, v_\infty \rangle$$

$$\geq \|v_\infty\|^2 + \underbrace{2 \langle v_n - v_\infty, v_\infty \rangle}_{\rightarrow 0}$$

Therefore

$$\liminf_{n \rightarrow \infty} \|v_n\|^2 \geq \|v_\infty\|^2$$

— 0 — 0 —

Theorem 2 (weak compactness of balls)

Assume that H is separable (actually we DO NOT NEED IT)

Let $\{v_n\} \subseteq H$ be a bounded sequence.

Then there exist an increasing sequence n_k of positive integers and $v_\infty \in H$ such that

$$v_{n_k} \longrightarrow v_\infty \text{ weakly in } H.$$

Proof. Let $M \in \mathbb{R}$ be such that $\|v_n\| \leq M$ for every $n \geq 1$.

Let $\{e_n\}$ be an orthonormal basis (Hilbert basis, complete orthonormal system).

Step 1 Let us consider e_1 . Then

$$|\langle v_n, e_1 \rangle| \leq \|v_n\| \leq M$$

Therefore $\{\langle v_n, e_1 \rangle\}$ admits a subsequence converging to some a_1 .
↑ sequence of numbers

Let us call N_1 the indices of the subseq.

Let us consider e_2 . Then

$$|\langle v_n, e_2 \rangle| \leq M \quad \forall n \in N_1$$

Therefore the PREVIOUS subsequence has a further subseq. that converges to some a_2 . Let $N_2 \subseteq N_1$ be the indices.

In this way we find $N \supseteq N_1 \supseteq N_2 \supseteq N_3 \supseteq \dots$

Key idea (diagonal argument): there exists an increasing sequence $\{m_k\}$ such that $m_k \in N_k$ for every $k \geq 1$.

(can be constructed by induction)

Claim:

$$\lim_{k \rightarrow \infty} \langle v_{m_k}, e_i \rangle = a_i \quad \forall i \geq 1$$

This is true because $m_k \in N_i \quad \forall k \geq i$

Step 2 The vector

$$v_\infty := \sum_{i=1}^{\infty} a_i e_i$$

is well defined. We need that $\sum_{i=1}^{\infty} a_i^2 < +\infty$

For every $m \in \mathbb{N}$ it is true that

FINITE SUM

$$\sum_{i=1}^m a_i^2 = \sum_{i=1}^m \lim_{k \rightarrow \infty} \langle v_{m_k}, e_i \rangle^2 \stackrel{\downarrow}{=} \lim_{k \rightarrow \infty} \sum_{i=1}^m \langle v_{m_k}, e_i \rangle^2$$

$$\leq \limsup_{k \rightarrow +\infty} \sum_{i=1}^{\infty} \langle v_{m_k}, e_i \rangle^2$$

$$= \limsup_{k \rightarrow +\infty} \|v_{m_k}\|^2 \leq M^2$$

Since m is arbitrary, the series converges.

Step 3 We have shown that

$$\lim_{k \rightarrow +\infty} \langle v_{m_k}, e_i \rangle = a_i = \langle v_{\infty}, e_i \rangle \quad \forall i \geq 1$$

This is enough to show that $v_{m_k} \rightarrow v_{\infty}$
 — o — o —

More general statement let $\{v_m\} \subseteq H$ be a bounded sequence.

Let us assume that

$$\langle v_m, v \rangle \rightarrow \langle v_{\infty}, v \rangle \quad \forall v \in W \subseteq H$$

$\uparrow$ subset

Then we can conclude that $v_m \rightarrow v_{\infty}$ provided that

$$\text{Clos}(\text{Span}(W)) = H$$

Proof Same as in the case where $W = \{e_n\}_{n \geq 1}$.
 — o — o —

Prop. (Normal convergence in Banach spaces)

Let V be a Banach space (complete normed vector space).

Let $\{v_m\} \subseteq V$ be a sequence.

Then

$$\sum_{n=1}^{\infty} \|v_n\| < +\infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} v_n \text{ converges in } V$$

Proof $S_m := v_1 + \dots + v_m$

$$\hat{S}_m := \|v_1\| + \dots + \|v_m\|$$

for every $m > n$ we have

$$\|S_m - S_n\| = \left\| \sum_{k=n+1}^m v_k \right\| \leq \sum_{k=n+1}^m \|v_k\| = \hat{S}_m - \hat{S}_n$$

$$\sum_{n=1}^{\infty} \|v_n\| < +\infty \iff \{\hat{S}_n\} \text{ is a Cauchy seq. in } \mathbb{R}$$

$$\implies \{S_n\} \text{ is a Cauchy seq. in } V$$

$$\iff \sum_{n=1}^{\infty} v_n \text{ converges in } V.$$

— o — o —