

Proof of Lemma 3 For every n let us write

$$v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n + r_n$$

$\uparrow$
remainder

claim: $\langle r_n, e_k \rangle = 0$ for $k = 1, \dots, n$

$$r_n = v - \langle v, e_1 \rangle e_1 - \dots - \langle v, e_n \rangle e_n$$

$$\begin{aligned} \langle r_n, e_k \rangle &= \langle v, e_k \rangle - \langle v, e_1 \rangle \langle e_1, e_k \rangle - \dots - \langle v, e_n \rangle \langle e_n, e_k \rangle \\ &= \langle v, e_k \rangle - \langle v, e_k \rangle = 0 \end{aligned}$$

Therefore $\|v\|^2 = \langle v, e_1 \rangle^2 + \dots + \langle v, e_n \rangle^2 + \|r_n\|^2$

$$\geq \sum_{k=1}^n \langle v, e_k \rangle^2$$

Since this is true for every $n \in \mathbb{N}$, the conclusion follows.

Proof of Theorem 2 The key point is proving that

$$v = \sum_{n=1}^{\infty} \langle v, e_n \rangle e_n$$

Convergence of the series. By Lemma 2

$$\sum_{n=1}^{\infty} \langle v, e_n \rangle e_n \text{ converges} \Leftrightarrow \sum_{n=1}^{\infty} \langle v, e_n \rangle^2 < +\infty, \text{ and this}$$

is true because of Lemma 3.

Let $\hat{v}$ be the sum of the series. By Lemma 4 we know that

$$\langle \hat{v}, e_n \rangle = \langle v, e_n \rangle \quad \forall n \geq 1 \quad \text{and therefore}$$

$$\langle \hat{v} - v, e_n \rangle = 0 \quad \forall n \geq 1.$$

Conclusion by Lemma 5.

Achtung! When preparing this exam, BEWARE of Thm 2.

Proof of Theorem 1

General case It is enough to consider a maximal orthonormal subset (Zorn lemma).

Separable case Let $\{v_n\} \subseteq H$ be a dense subset.

Then there exists a subsequence $\{v_{n_k}\}$ such that

$$\dim \text{Span} \{v_{n_1}, \dots, v_{n_k}\} = k \quad (\text{easy induction})$$

"Span $\{v_1, v_2, \dots, v_{n_k}\}$ [Added after video]

One can prove that $\text{Span} \{v_n\} = \text{Span} \{v_{n_k}\}$

Apply Gram-Schmidt process to $\{v_{n_k}\}$.

↑ same procedure as in $\mathbb{R}^d$.

Remark The process stops on the road if the algebraic dimension of H is finite.

— o — o —

WEAK CONVERGENCE

Def. Let H be a Hilbert space, and let $\{v_m\} \subseteq H$ be a sequence, and let $v_\infty \in H$.

We say that v_m converges weakly to v_∞ , and we write

$$v_m \longrightarrow v_\infty$$

$$\text{if } \langle v_m, v \rangle \longrightarrow \langle v_\infty, v \rangle \quad \forall v \in H$$

$\uparrow$
conv. of numbers

Trivial properties

① Uniqueness of the limit

Assume $v_m \longrightarrow v_\infty$ and $v_m \longrightarrow \hat{v}_\infty$. Then

$$\langle v_\infty, v \rangle = \lim_{m \rightarrow \infty} \langle v_m, v \rangle = \langle \hat{v}_\infty, v \rangle \quad \forall v \in V$$

$$\leadsto \langle v_\infty - \hat{v}_\infty, v \rangle = 0 \quad \forall v \in V \quad \text{Use } v = v_\infty - \hat{v}_\infty \quad \ddot{\smile}$$

② Stability by subsequences: if $v_m \longrightarrow v_\infty$, then $v_{m_k} \longrightarrow v_\infty$

③ Linearity: if $v_m \longrightarrow v_\infty$ and $w_m \longrightarrow w_\infty$, then

$$\alpha v_m + \beta w_m \longrightarrow \alpha v_\infty + \beta w_\infty \quad \forall (\alpha, \beta) \in \mathbb{R}^2$$

④ Strong implies weak: if $v_m \longrightarrow v_\infty$, then $v_m \longrightarrow v_\infty$

(continuity of scalar prod.)

⑤ If $\dim H < +\infty$ then $u_n \rightarrow u_\infty \Leftrightarrow u_n \rightharpoonup u_\infty$

Let $\{e_1, \dots, e_d\}$ be an orthonormal basis. Then

$$u_n \rightarrow u_\infty \Rightarrow \langle u_n, e_i \rangle \rightarrow \langle u_\infty, e_i \rangle \quad \forall i \in \{1, \dots, d\}$$

And in finite dim conv. of components is equiv. to conv. of vectors

$$u_n = \sum_{i=1}^d \langle u_n, e_i \rangle e_i \xrightarrow{n \rightarrow \infty} \sum_{i=1}^d \langle u_\infty, e_i \rangle e_i = u_\infty$$

Achtung! The limit commutes with finite sums, not with series.

⑥ If $\dim H = +\infty$, then there are sequences that converge weakly but not strongly.

Example Let $\{e_n\}_{n \geq 1}$ be an orthonormal system (it always exists if $\dim H = +\infty$)

Then

$$e_n \rightarrow 0$$

$$\text{but of course } e_n \not\rightarrow 0 \quad (\|e_n - 0\| = 1)$$

In order to prove weak convergence we need to show that

$$\langle e_n, v \rangle \rightarrow \langle 0, v \rangle = 0 \quad \forall v \in H$$

This is true because $\sum_{n=1}^{\infty} \langle e_n, v \rangle^2 < +\infty$ (Bessel)

$$\text{Conv. series} \Rightarrow \langle e_n, v \rangle \rightarrow 0.$$

BIG ISSUE

If $\dim H = +\infty$, then closed balls are NOT STRONGLY COMPACT

$\{e_n\} \subseteq$ Ball of radius 1, but it admits no conv. subseq.

$$\|e_n - e_m\|^2 = \|e_n\|^2 + \|e_m\|^2 = 2 \quad \forall m \neq n$$

So $\{e_n\}$ does NOT ADMIT Cauchy subsequences.

— o — o —

In $\mathbb{R}^d$ convergence of vectors is equiv. to conv. of components

In a general Hilbert space

→ convergence of vectors is strong conv.

→ " of components is weak conv.

Prop. Let H be a Hilbert space

Let $\{e_n\}_{n \geq 1}$ be a Hilbert basis

Let $\{v_n\} \subseteq H$ be a sequence, and let $v_\infty \in H$

Let us assume that $\{v_n\}$ is bounded, namely

$$\exists M \in \mathbb{R} : \|v_n\| \leq M \quad \forall n \geq 1$$

Then $v_n \rightarrow v_\infty$ if and only if

$$\langle v_n, e_k \rangle \longrightarrow \langle v_\infty, e_k \rangle \quad \forall k \geq 1.$$

Proof The nontrivial implication is $\Leftarrow$

Idea: $|\langle v_\infty, v \rangle - \langle v_m, v \rangle| \leq |\langle v_\infty, v \rangle - \langle v_\infty, v_\varepsilon \rangle|$ ①

$$+ |\langle v_\infty, v_\varepsilon \rangle - \langle v_m, v_\varepsilon \rangle|$$
 ②

$$+ |\langle v_m, v_\varepsilon \rangle - \langle v_m, v \rangle|$$
 ③

① $\leq \|v_\infty\| \cdot \|v - v_\varepsilon\| \rightsquigarrow$ small if $v_\varepsilon \sim v$

② is small if m is large and $v_\varepsilon \in \text{Span}\{e_k\}$

③ $\leq \|v_m\| \cdot \|v - v_\varepsilon\| \rightsquigarrow$ small if $v_\varepsilon \sim v$
 $\leq M$

Formally

→ let $\varepsilon > 0$ be fixed

→ Choose $v_\varepsilon \in \text{Span}\{e_k\}$ (FINITE LIN COMB) s.t.

$$\|v_\infty\| \cdot \|v - v_\varepsilon\| \leq \frac{\varepsilon}{4}$$

$$M \cdot \|v - v_\varepsilon\| \leq \frac{\varepsilon}{4}$$

→ After choosing v_ε , choose m_0 s.t.

$$|\langle v_\infty, v_\varepsilon \rangle - \langle v_m, v_\varepsilon \rangle| \leq \frac{\varepsilon}{4} \quad \forall m \geq m_0$$

(this is possible because v_ε is FINITE lin. comb. of e_k 's)

→ As before conclude that

$$|\langle v_m, v \rangle - \langle v_\infty, v \rangle| \leq \frac{3}{4} \varepsilon \quad \forall m \geq m_0$$

This implies that $\langle v_m, v \rangle \rightarrow \langle v_\infty, v \rangle$.

It is essential that we assume $\{v_m\}$ to be bounded.

just consider $v_m = m e_1$. Claim (exercise)

$$\langle v_m, e_k \rangle \rightarrow 0 \quad \forall k \geq 1 \quad \text{but} \quad v_m \not\rightarrow 0.$$