

Vector spaces + structure

$\mathbb{R}^d$

Hilbert spaces (everything works, with some notable exceptions)

Banach spaces (nothing works, ")

Normed spaces

Locally convex top. vect. spaces (as Banach spaces, but with more work)

Topological vector spaces

HILBERT SPACE **Vector space** + positive definite **Scalar product**
 scal. prod. $\rightsquigarrow$ norm $\rightsquigarrow$ distance $\rightsquigarrow$ notion of convergence
Complete metric space wrt this distance

HILBERT BASIS Let H be a Hilbert space.

A subset $\{e_i\}_{i \in I}$ (I is an index set) is **orthonormal** if

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

A Hilbert basis is an orthonormal subset $\{e_i\}_{i \in I}$ such that

$\text{Span } \{e_i\}$ is dense in H

$\uparrow$ set of FINITE linear combinations of $\{e_i\}$

COMPONENTS OF A VECTOR

Let H be a Hilbert space, let $v \in H$

let $\{e_i\}_{i \in I}$ be orthonormal set

Then $v_i := \langle v, e_i \rangle$ is called the component of v wrt $\{e_i\}$

$\uparrow$
real number

Theorem 1 Every Hilbert space H admits a Hilbert basis $\{e_i\}_{i \in I}$

If H is **SEPARABLE**, namely there exists a COUNTABLE dense subset, then I is either FINITE or COUNTABLE.

Theorem 2 Let H be a Hilbert space.

Let us assume that there exists a countable Hilbert basis $\{e_n\}_{n \geq 1}$

Let $v \in H$, and let $v_n = \langle v, e_n \rangle$ be its components.

Then the following facts are true

$$v = \sum_{n=1}^{\infty} v_n e_n = \sum_{n=1}^{\infty} \langle v, e_n \rangle e_n$$

$$\|v\|^2 = \sum_{n=1}^{\infty} v_n^2 = \sum_{n=1}^{\infty} \langle v, e_n \rangle^2 \quad (\text{PITAGORA'S THM})$$

$$\langle v, w \rangle = \sum_{n=1}^{\infty} \langle v, e_n \rangle \cdot \langle w, e_n \rangle$$

In particular, all the series are convergent.

Remark Due to thm 2, we can identify any vector $v \in H$ with the sequence $\{v_n\}$ of its components.

In a brutal sense, a separable Hilbert space is " $\mathbb{R}^\infty$ ".

Lemma 1 (Continuity of norm and scalar product)

Let us assume that $v_n \rightarrow v_\infty$ in H and $w_n \rightarrow w_\infty$ in H .

Then

$$\|v_n\| \rightarrow \|v_\infty\| \quad \text{and} \quad \langle v_n, w_n \rangle \rightarrow \langle v_\infty, w_\infty \rangle$$

Lemma 2 (Convergence of Hilbert series)

Let $\{e_n\} \subseteq H$ be an orthonormal set

Let $\{a_n\}$ be a sequence of real numbers.

Then

$$\sum_{n=1}^{\infty} a_n e_n \text{ converges} \iff \sum_{n=1}^{\infty} a_n^2 \text{ converges}$$

$\uparrow$ series of vectors $\uparrow$ series of numbers

Lemma 3 (Bessel inequality)

Let $\{e_n\} \subseteq H$ be an orthonormal system. Let $v \in H$ be a vector.

Then

$$\sum_{n=1}^{\infty} \langle v, e_n \rangle^2 \leq \|v\|^2$$

Lemma 4 (Properties of Hilbert series)

Let $\{e_n\}_{n \geq 1} \subseteq H$ be an orthonormal system.

Let $\{a_n\} \subseteq \mathbb{R}$ be a sequence such that

$$\sum_{n=1}^{\infty} a_n^2 < +\infty$$

From Lemma 2 we know that $v := \sum_{n=1}^{\infty} a_n e_n \in H$ is well-def.

Then $\langle v, e_n \rangle = a_n \quad \forall n \geq 1$ and $\|v\|^2 = \sum_{n=1}^{\infty} a_n^2$

Lemma 5 (Characterization of the null vector)

Assume that $\{e_n\}_{n \geq 1} \subseteq H$ is a Hilbert basis. Let $v \in H$.

Assume that

$$\langle v, e_n \rangle = 0 \quad \forall n \geq 1$$

Then $v = 0$.

Proof of Lemma 1

$$| \|u_\infty\| - \|u_m\| | \leq \|u_\infty - u_m\| \quad (\text{triangular ineq.})$$

$$| \langle u_\infty, w_\infty \rangle - \langle u_m, w_m \rangle | \leq | \langle u_\infty, w_\infty \rangle - \langle u_\infty, w_m \rangle | + | \langle u_\infty, w_m \rangle - \langle u_m, w_m \rangle |$$

$$\leq \underbrace{\|u_\infty\|}_{\substack{\uparrow \\ \text{c.s.}}} \cdot \underbrace{\|w_\infty - w_m\|}_{\substack{\downarrow \\ 0}} + \underbrace{\|w_m\|}_{\substack{\downarrow \\ \text{unif. bounded, namely } \|w_m\| \leq M}} \cdot \underbrace{\|u_\infty - u_m\|}_{\substack{\downarrow \\ 0}}$$

unif. bounded, namely $\|w_m\| \leq M$

Sublemma: any convergent sequence is bounded.

Proof of Lemma 2

$$S_m := \sum_{k=1}^m a_k e_k \quad \hat{S}_m := \sum_{k=1}^m a_k^2$$

For every $m > n$

$$\|S_m - S_n\|^2 \underset{\substack{\uparrow \\ \text{defn. of } S_n}}{=} \left\| \sum_{k=n+1}^m a_k e_k \right\|^2 = \sum_{k=n+1}^m a_k^2 \underset{\substack{\uparrow \\ \{e_k\} \text{ orthonorm. defn. of } \hat{S}_n}}{=} \hat{S}_m - \hat{S}_n$$

$$\sum_{n=1}^{\infty} a_n e_n \text{ conv.} \Leftrightarrow \{S_n\} \text{ is a Cauchy seq. in } H$$

$$\Leftrightarrow \{\hat{S}_n\} \text{ is a Cauchy seq. in } \mathbb{R}$$

$$\Leftrightarrow \sum_{n=1}^{\infty} a_n^2 \text{ converges.}$$

Proof of Lemma 4

$$\langle u, e_k \rangle = \lim_{n \rightarrow +\infty} \underbrace{\langle S_n, e_k \rangle}_{= a_k \text{ for every } n \geq k} = a_k$$

$$\|v\|^2 = \lim_{n \rightarrow +\infty} \|S_n\|^2 = \lim_{n \rightarrow +\infty} \sum_{k=1}^n a_k^2 = \sum_{k=1}^{\infty} a_k^2.$$

Proof of Lemma 5 Let us fix $\varepsilon > 0$.

There exists $v_\varepsilon \in \text{Span}\{e_n\}$ s.t. $\|v - v_\varepsilon\|^2 \leq \varepsilon$
(because Span is dense).

$$\varepsilon \geq \|v - v_\varepsilon\|^2 = \|v\|^2 + \|v_\varepsilon\|^2 - 2 \underbrace{\langle v, v_\varepsilon \rangle}_{=0}$$

because v_ε is a FINITE
lin. comb. of e_i 's

$$\geq \|v\|^2$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\|v\| = 0$, and hence
 $v = 0$.

(we need $\langle \cdot, \cdot \rangle$ to be pos. def.)

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