

Horizontal variation (inner variation)

Vertical variation : $u_0(x) + t v(x)$

Horizontal variation : $u_0(x + t v(x))$

Assume $u_0 \in C^1([a, b])$. Assume that $v \in C_c^\infty((a, b))$.

Then there exists $\delta > 0$ s.t.

$$a \leq x + t v(x) \leq b \quad \forall x \in [a, b] \quad \forall t \in [-\delta, \delta]$$

and therefore $u_0(x + t v(x))$ is a good competitor.

Two cases where it is fundamental

→ case where u_0 has values in S^1 or some manifold

→ geometric problems, where the unknown is a set identified with its characteristic function.

Let us compute the first variation of

$$F(u) = \int_a^b L(x, u(x), u'(x)) dx$$

wrt to a hor. var.

$$\varphi(t) := \int_a^b L(x, u(x + t v(x)), \underbrace{u'(x + t v(x)) (1 + t v'(x))}_{\text{deriv. wrt } x \text{ of } u(x + t v(x))}) dx$$

$$\begin{aligned} \varphi'(t) = & \int_a^b L_s(\dots) u(x + t v(x)) v(x) + \\ & + L_p(\dots) [u(x + t v(x)) v(x) (1 + t v'(x)) + u(x + t v(x)) v'(x)] \end{aligned}$$

and therefore

$$\begin{aligned}\varphi'(0) &= \int_a^b \left\{ L_s(x, u(x), \dot{u}(x)) \dot{u}(x) v(x) + L_p(x, u(x), \dot{u}(x)) \right. \\ &\quad \left. [\ddot{u}(x) v(x) + \dot{u}(x) \dot{v}(x)] \right\} dx \\ &\quad \quad \quad [\dot{u}(x) v(x)]' \\ &= \int_a^b \left\{ L_s(x, u(x), \dot{u}(x)) - [L_p(x, u(x), \dot{u}(x))]' \right\} \dot{u}(x) v(x) dx \\ &\quad + \text{boundary terms}\end{aligned}$$

If we ignore boundary terms, FLCV implies that

$$\left\{ L_s(\dots) - [L_p(\dots)]' \right\} \dot{u}(x) = 0$$

$\uparrow$
previous ELE
 $\uparrow$
we should have expected this $\ddot{u}$

If $u_0(x)$ satisfies this new ELE and $\dot{u}_0(x)$ vanishes only in a finite number of points, then $u_0(x)$ satisfies the classical ELE.

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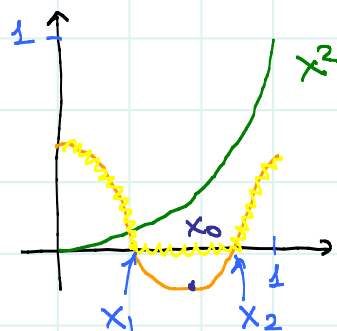
TRUNCATION ARGUMENTS VS MAXIMUM PRINCIPLES

Example $\min \left\{ \int_0^1 [\dot{u}^2 + (u - x^2)^2] dx : u \in C^1([0, 1]) \right\}$

Claim: a unique minimizer $u_0(x)$ exists and $0 \leq u_0(x) \leq 1$.

There is NO REASON for $u_0(x)$ to be > 1 or < 0 .

Assume that $u(x_0) < 0$. Then we obtain a better competitor by TRUNCATING $u(x)$



Let $u(x_0) < 0$. Let

$$x_1 := \sup \{ x \in [0, x_0] : u(x) = 0 \}$$

$$x_2 := \inf \{ x \in [x_0, 1] : u(x) = 0 \}$$

Then define

$$\hat{u}(x) := \begin{cases} u(x) & \text{if } x \in [0, x_1] \\ 0 & \text{if } x \in [x_1, x_2] \\ u(x) & \text{if } x \in [x_2, 1] \end{cases}$$

and observe that

$$F(\hat{u}) < F(u)$$

↑
we save on both terms in $F(u)$

Complain #1: $\hat{u}$ is not C^4 , BUT it is piecewise C^4 , and I can keep the strict inequality even with a smooth connection in x_1 and x_2

Complain #2: x_1 and x_2 , as defined, could not exist. But in that case it is even easier (use 0 instead of x_1 and 1 instead of x_2).

Remark If $u \in C^4$, then $\max \{ u(x), 0 \}$ is NOT necessarily piecewise C^4 .

In this case ELE is

$$(2\ddot{u})' = 2(u - x^2) \quad \leadsto \quad \ddot{u} = u - x^2$$

Assume that $\min \{ u_0(x) : x \in [0, 1] \} < 0$. Let $x_0 \in (0, 1)$ be a min. point of $u_0(x)$.

Then necessarily $\ddot{u}_0(x_0) \geq 0$. On the other hand

$$\ddot{u}_0(x_0) = u_0(x_0) - x_0^2 \leq u_0(x_0) < 0$$

Complain: it could be $x_0 = 0$ or $x_0 = 1$.

But even in this case the same argument applies because again $\ddot{u}_0(x_0) \geq 0$ (please consider that this is not trivial: for example $\log x$ has second derivative which is negative and min. at the boundary in every interval)

BUT this is true if we know that $u'(x_0) = 0$.

In this case we know that $u'(x_0)$ because of NBC.