

Setting: S is a set, $F: S \rightarrow \mathbb{R}$ is a function, $s_0 \in S$
 How to prove that s_0 is a min point?

Two main strategies: $\rightarrow$ via CONVEXITY
 $\rightarrow$ via AUXILIARY FUNCTIONAL

MINIMALITY VIA CONVEXITY

Main example

$$\min \left\{ \int_a^b \varphi(u'(x)) dx : u \in C^1([a, b]), \right. \\ \left. u(a) = A, u(b) = B \right\}$$

If $u_0(x)$ is a minimizer, then $[\varphi'(u'_0(x))] = 0$, namely

$$\varphi''(u'_0(x)) u''_0(x) = 0 \quad \text{ELE}$$

Let $u_0(x)$ denote the straight line that satisfies DBC.
 Claim: $u_0(x)$ is the unique minimizer.

Prop. The claim is true if φ is CONVEX.

If φ is strictly convex, then the minimizer is unique.

Proof. Let us consider the inequality

$$(*) \quad \varphi(a+b) \geq \varphi(a) + \varphi'(a)b \quad \forall a \in \mathbb{R} \quad \forall b \in \mathbb{R}$$

Taylor-Lagrange: $\varphi(a+b) = \varphi(a) + \varphi'(a)b + \underbrace{\frac{1}{2} \varphi''(\xi) b^2}_{\geq 0}$

$\uparrow$
somewhere

$$0 \uparrow u \{a, b\}$$

$$v(b) - v(a) = 0$$

If φ is strictly convex, then (*) is strict whenever $b \neq 0$, namely (**) is strict whenever $v'(x) \neq 0$.

Remark If φ is a convex function, then for every $a \in \mathbb{R}$ there exists $\alpha \in \mathbb{R}$ such that

$$\varphi(a+b) \geq \varphi(a) + \alpha b \quad \forall b \in \mathbb{R}$$



left and right derivative

Theorem Let us consider a Lagrangian $L(x, s, p)$. Assume that, for every $x \in [a, b]$, the function

$$(s, p) \mapsto L(x, s, p)$$

TS convex in the pair (s, p) (convexity in $\mathbb{R}^2$).

Then any solution to ELE is a minimizer, unique if L is strictly convex.

Proof Start from the inequality

$$L(x, s_0 + s_1, p_0 + p_1) \geq L(x, s_0, p_0) + L_s(x, s_0, p_0) s_1 + L_p(x, s_0, p_0) p_1$$

For every $x \in [a, b]$ and $(s_0, s_1, p_0, p_1) \in \mathbb{R}^4$.

Let $u_0(x)$ be a solution to ELE (just in the first int. form)

$$\text{Then } u(x) = u_0(x) + v(x)$$

$$\begin{aligned} F(u) &= \int_a^b L(x, \underset{\uparrow s_0}{u_0(x)} + \underset{\uparrow s_1}{v(x)}, \underset{\uparrow p_0}{u'_0(x)} + \underset{\uparrow p_1}{v'(x)}) dx \\ &\geq F(u_0) + \underbrace{\int_a^b \{L_s(x, u_0(x), u'_0(x)) v(x) + L_p(\dots) v'(x)\} dx}_{= 0 \text{ because of ELE}} \end{aligned}$$

If we have strict convexity, we argue as before.
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MINIMALITY VIA AUXILIARY FUNCTIONAL

↑ "OSCUATING"

Lemma S is a set, $F: S \rightarrow \mathbb{R}$, $G: S \rightarrow \mathbb{R}$, $s_0 \in S$.

Assume that

$$(i) F(x) \geq G(x) \quad \forall x \in S$$

$$(ii) G(x) \geq G(s_0) \quad \forall x \in S$$

$$(iii) G(s_0) = F(s_0)$$

Then s_0 is a min. point of F .

Proof

$$F(x) \geq \underset{\uparrow (i)}{G(x)} \geq \underset{\uparrow (ii)}{G(s_0)} = \underset{\uparrow (iii)}{F(s_0)}$$

$$\forall x \in S \quad \text{☺}$$

Easy but not trivial example

$$F(u) = \int_0^1 (u'(x)^2 - 1)^2 dx$$

$$\min \{ F(u) : u \in C^1(\dots), \\ u(0) = 0, u(1) = 4 \}$$

Claim: $u_0(x)$ is the unique minimizer

Consider $G(u) := \int_0^1 \hat{\varphi}(u'(x)) dx$

It is true that (i) $F(u) \geq G(u)$ for every admissible u
 $\uparrow$
 $\varphi(p) \geq \hat{\varphi}(p)$

(ii) $G(u) \geq G(u_0)$ (because $\hat{\varphi}$ is convex)

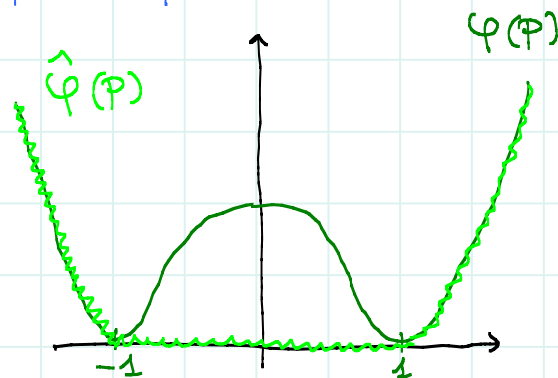
(iii) $G(u_0) = F(u_0)$ (because $\hat{\varphi}(4) = \varphi(4)$)

From the lemma we conclude that $u_0(x)$ minimizes F .

Actually it is the unique min. because $G(u) \geq G(u_0)$ is strict whenever $u(x) \neq u_0(x)$ (because the graph of $\hat{\varphi}$ is strictly above the tangent line in $p = 4$).

The same argument applies if we replace 4 with any B with $|B| \geq 1$.

$$\varphi(p) = (p^2 - 1)^2$$



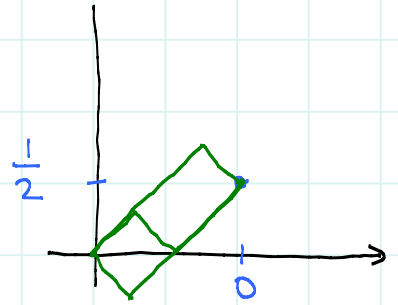
DOUBLE WELL POTENTIAL

$\uparrow$
POZZO
(BRÜNNEN)

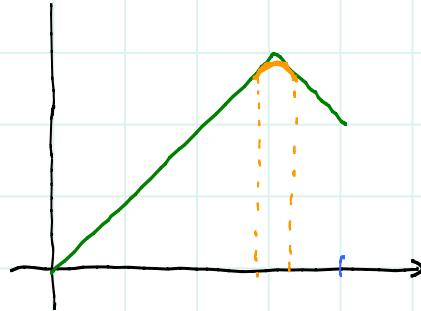
$$\min \{ F(u) : u \in C^1(\dots), u(0) = 0, u(1) = \frac{1}{2} \}$$

The $\inf = 0$ and $\min$ DOES NOT EXIST.

A $\inf$ -izing sequence need to have derivative ± 1 "almost everywhere"

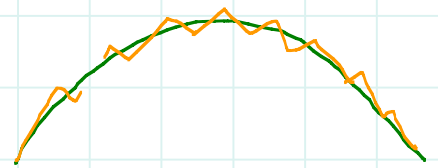


The cost of a smooth connection is at most the length of the interval where it takes place.



$$\min \left\{ \int_0^1 \{ (\dot{u}(x)^2 - 1)^2 + (u - \sin x)^2 \} dx : u \in C^1 \right\}$$

Again $\inf = 0$ because I can approx $\sin x$ with u which has derivative ± 1 .



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