

FIRST VARIATION FOR MULTIPLE INTEGRALS

$$F(u) = \int_{\Omega} L(x, u(x), \nabla u(x)) dx$$

where $\Omega \subseteq \mathbb{R}^d$ is a bounded open set with "smooth" boundary and

$$L: \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$$

$$L(x, s, p)$$

↑ vector

Gauss-Green formula (integration by parts for multiple int.)

$$\int_{\Omega} f \cdot \operatorname{div} \vec{E} dx = \int_{\partial\Omega} f \langle \vec{E}, \vec{n} \rangle d\sigma - \int_{\Omega} \langle \nabla f, \vec{E} \rangle dx$$

$$f: \Omega \rightarrow \mathbb{R}$$

$$\vec{E}: \Omega \rightarrow \mathbb{R}^d$$

↑
outward unit
normal to $\partial\Omega$

integral in
(d-1) dimensions

In the special case where $\vec{E} = \nabla g$ for some $g: \Omega \rightarrow \mathbb{R}$ the formula becomes

$$\int_{\Omega} f \Delta g dx = \int_{\partial\Omega} f \langle \nabla g, \vec{n} \rangle d\sigma - \int_{\Omega} \langle \nabla f, \nabla g \rangle dx$$

$$\int_{\partial\Omega} f \frac{\partial g}{\partial \vec{n}} d\sigma$$

↑ derivative of g
in the direction $\vec{n}$

The formula requires some "smoothness" of the boundary $\partial\Omega$.
For example $\partial\Omega$ piecewise C^1 is enough.

Come back to $F(u)$ defined above. let us start with the special case

$$F(u) = \int_{\Omega} |\nabla u(x)|^2 dx$$

↑ norm of the gradient

(DIRICHLET
INTEGRAL)

$$\begin{aligned}\varphi(t) &:= F(u+tv) = \int_{\Omega} |\nabla u(x) + t \nabla v(x)|^2 dx \\ &= \int_{\Omega} |\nabla u(x)|^2 dx + 2t \int_{\Omega} \langle \nabla u(x), \nabla v(x) \rangle dx \\ &\quad + t^2 \int_{\Omega} |\nabla v(x)|^2 dx\end{aligned}$$

$$\delta F(u, v) = \varphi'(0) = 2 \int_{\Omega} \langle \nabla u(x), \nabla v(x) \rangle dx$$

FIRST INT.
FORM

If we have more regularity, we int. by parts and we obtain

$$\delta F(u, v) = -2 \int_{\Omega} \Delta u(x) v(x) dx + 2 \int_{\partial \Omega} \frac{\partial u(x)}{\partial \vec{n}} \cdot v(x) d\sigma$$

SECOND INTEGRAL FORM

If u is a DLM wrt every $v \in C_c^\infty(\Omega)$, then FLCV implies that

$$\Delta u(x) = 0 \quad \forall x \in \Omega$$

ELE in diff. form

If we can use functions v which are not 0 at the boundary we obtain "on the road" NBC of the form

$$\frac{\partial u}{\partial \vec{n}} = 0 \quad \text{at } \partial \Omega$$

Consider now a more general

$$F(u) = \int_{\Omega} L(x, u(x), \nabla u(x)) dx$$

$$\varphi(t) = \int_{\Omega} L(x, u(x) + t v(x), \nabla u(x) + t \nabla v(x)) dx$$

$$\delta F(u, v) = \varphi'(0) = \int_{\Omega} \{ L_s(x, u(x), \nabla u(x)) v(x) + \\ + \langle \nabla_p L(x, u(x), \nabla u(x)), \nabla v(x) \rangle \} dx$$

After integration by parts we obtain

$$\delta F(u, v) = \int_{\Omega} \{ L_s(x, u(x), \nabla u(x)) - \operatorname{div}(\nabla_p L(x, u(x), \nabla u(x))) \} v(x) dx \\ + \int_{\partial\Omega} v(x) \langle \nabla_p L(x, u(x), \nabla u(x)), \vec{n} \rangle d\sigma$$

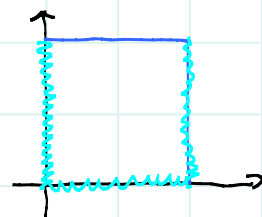
If u is DLM wrt every $v \in C_c^\infty(\Omega)$ (and both u and L are regular enough)

then u satisfies

$$\operatorname{div}(\nabla_p L(x, u(x), \nabla u(x))) = L_s(x, u(x), \nabla u(x))$$

Example $\Omega = (0,1)^2$ $F(u) = \int_{\Omega} \{ |\nabla u(x)|^2 - u(x) \} dx$

Problem: minimize $F(u)$ over all u 's of class C^1 which are 0 on three sides of the boundary



Assume that $u_0(x,y)$ is a min. of class C^2 . Then nec.

$$2 \Delta u = -1$$

$$L(x, s, p_1, p_2) = -s + p_1^2 + p_2^2$$
$$(L_{p_1}(\dots), L_{p_2}(\dots)) = (2p_1, 2p_2)$$

$$\nabla_p L(u, \nabla u) = 2(u_x, u_y)$$

$\mu \equiv 0$ on the 3 sides

DBC from the very beginning

$M_y = 0$ on the 4th side

NBC born on the road

$$\mathbb{X} = \{u \in C^1(\bar{\Omega}) : u \equiv 0 \text{ on the 3 sides}\}$$

$$V = \{ \dots \}$$

When we compute the 2nd int. form of $\delta F(u, v)$, the only boundary term that survives is

$$\int_{\substack{\uparrow \\ \text{4th side}}} S v(x, y) n_y(x, y) d\sigma = \int_0^1 v(x, \underbrace{1}_{\substack{\uparrow \\ \text{param. of 4th side}}}) n_y(x, 1) dx$$

FLCV on the 4th side $\Rightarrow u_y(x, 1) = 0 \quad \forall x \in [0, 1]$.

Example $F(u) = \int_{\mathbb{Q}} (y^2 \underbrace{u_x^2}_{\text{square}} + u_y^2) dx dy$

Compute ELE $L(x, y, s, p_1, p_2) = y^2 p_1^2 + p_2^2$

$$L_S = 0 \quad (L_{p_1}, L_{p_2}) = (2y^2 p_1, 2p_2) = (2y^2 u_x, 2u_y)$$

$$2 \operatorname{div} (y^2 u_x, u_y) = 0 \quad \text{an ELE}$$

$$y^2 u_{xx} + u_{yy} = 0 \quad 2x u_x + x^2 u_{xx} + u_{yy} = 0$$