

ELE in general Setting:

$$\int_a^b L(x, u(x), \dot{u}(x)) dx = F(u)$$

where $L: [a, b] \times \underset{x}{\mathbb{R}} \times \underset{p}{\mathbb{R}} \rightarrow \mathbb{R} \quad L(x, s, p)$

Goal: compute the first variation. We consider

$$\varphi(t) := F(u + tv) = \int_a^b L(x, u(x) + tv(x), \dot{u}(x) + t\dot{v}(x)) dx$$

$$\varphi'(t) = \int_a^b \{ L_s(x, u + tv, \dot{u} + t\dot{v}) v + L_p(\dots) \dot{v} \} dx$$

$$\delta F(u, v) := \varphi'(0) =$$

FIRST INTEGRAL FORM

$$\int_a^b \{ L_s(x, u(x), \dot{u}(x)) v(x) + L_p(x, u(x), \dot{u}(x)) \dot{v}(x) \} dx$$

Sufficient conditions: $\rightarrow u$ and v in $C^1([a, b])$
 $\rightarrow L$ of class C^1 (we just need L_s and L_p)

If we integrate by parts, we obtain

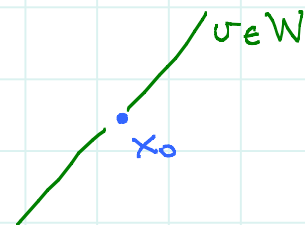
SECOND INTEGRAL FORM

$$\delta F(u, v) = \int_a^b \left\{ L_s(x, u(x), \dot{u}(x)) - \frac{d}{dx} L_p(x, u(x), \dot{u}(x)) \right\} v(x) dx + [L_p(x, u(x), \dot{u}(x)) v(x)]_{x=a}^{x=b}$$

Sufficient conditions: $\rightarrow u \in C^2([a,b])$
 $\rightarrow v \in C^1([a,b])$
 $\rightarrow L$ of class C^2 (L_p of class C^1)

Of course, if u is a minimizer, then $\delta F(u,v) = 0$ for every "admissible" v

Def. Let X be an affine space with vect. space V . Let $F: X \rightarrow \mathbb{R}$ be a functional, let $W \subseteq V$ be a vect. subspace, let $x_0 \in X$. We say that x_0 is a directional local minimum wrt W if, for every $v \in W$, the function $t \rightarrow F(x_0 + tv)$ has a loc. min. in $t=0$.



Remark x_0 min. point in $X \Rightarrow x_0$ is a DLM for every $W \subseteq V$
The converse is not true even if $W = V = X = \mathbb{R}^2$

Coming back to integral functionals, let us assume that u_0 is a DLM with $W = C_c^\infty((a,b))$.

Then we deduce that

$$\frac{d}{dx} L_p(x, u_0(x), \dot{u}_0(x)) = L_s(x, u_0(x), \dot{u}_0(x))$$

DIFF. FORM
OF ELE

Suff. conditions: the same as in 2nd int. form.

The derivative in the LHS is a total derivative. If we expand

$$L_{px}(\dots) + L_{ps}(\dots) \dot{u}_0 + L_{pp}(\dots) \ddot{u}_0 = L_s(\dots)$$

If $L_{pp}(x, s, p) > 0$ for every admissible values of the parameters, then ELE can be written as

$$\ddot{u}_0(x) = \Phi(x, u_0(x), \dot{u}_0(x))$$

for a suitable Φ . ($L_{pp} > 0$ is stronger than strict convexity)

If $u_0(x)$ is DLM with respect to a class W that contains $C_c^\infty(a, b)$ but also functions with $v(a) = 0$ and $v(b) = 1$, then a NBC appears of the form

$$L_p(b, u_0(b), \dot{u}_0(b)) = 0 \quad \text{NBC}$$

The same if $v(a) = 1$ and $v(b) = 0$.

ELE in DBR form

$$\int_a^b \{ L_s(x, u(x), \dot{u}(x)) v(x) + L_p(x, u(x), \dot{u}(x)) \dot{v}(x) \} dx$$

Let us set $\hat{L}(x) := \int_a^x L_s(t, u(t), \dot{u}(t)) dt$

If we integrate by parts the first term we obtain

$$\int_a^b \{ -\hat{L}(x) + L_p(x, u(x), \dot{u}(x)) \} \dot{v}(x) dx + [\hat{L}(x) v(x)]_{x=a}^{x=b}$$

Again, if $u(x)$ is DLM with $W = C_c^\infty(a, b)$ then we apply DBR lemma and we obtain

$$L_p(x, u(x), \dot{u}(x)) = c + \int_a^x L_s(t, u(t), \dot{u}(t)) dt$$

↑
free parameter

Remark The same can be obtained by integrating the diff. form, BUT the diff. form requires 2nd order deriv. of u and L , while here we used just 1st order derivative.

Remark DBR Lemma is trivial if we could int. by parts

$$\int_a^b u(x) \dot{\varphi}(x) dx = 0 \quad \rightsquigarrow \quad \int_a^b \dot{u}(x) \varphi(x) dx = 0$$

$$\rightsquigarrow \dot{u}(x) \equiv 0 \quad \rightsquigarrow \quad u(x) \equiv \text{constant}$$

— 0 — 0 —

ELE in the autonomous case Assume that $L(s, p)$ does NOT depend on x .

Start with ELE

$$[L_p(u, \dot{u})]' = L_s(u, \dot{u})$$

Multiply by $\ddot{u}$:

$$[L_p(u, \dot{u})]' \ddot{u} = L_s(u, \dot{u}) \ddot{u}$$

$$[L_p(u, \dot{u}) \ddot{u}]' - L_p(u, \dot{u}) \ddot{u}$$

and therefore

$$[L_p(u, \dot{u}) \ddot{u}]' = L_p(u, \dot{u}) \ddot{u} + L_s(u, \dot{u}) \ddot{u} = [L(u, \dot{u})]'$$

and this implies that

$$L_p(u(x), \dot{u}(x)) \ddot{u}(x) = L(u(x), \dot{u}(x)) + c$$

First order diff. equ. in u with a free parameter

More general situation

$$F(u) = \int_a^b L(x, u, \dot{u}, \ddot{u}) dx \quad L(x, s, p_1, p_2)$$

$$\delta F(u, v) = \int_a^b \{ L_s(\dots) v + L_{p_1}(\dots) \dot{v} + L_{p_2}(\dots) \ddot{v} \} dx$$

Assume $v \in C_c^\infty(a, b)$. Int. by parts we obtain

$$\delta F(u, v) = \int_a^b \underbrace{\{ L_s(\dots) - (L_{p_1}(\dots))' + (L_{p_2}(\dots))'' \}}_{=0} v(x) dx$$

$= 0 \leadsto$ ELE in diff. form

(equation of order 4)

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