

GENESIS OF BOUNDARY CONDITIONS

$$F(u) = \int_0^2 \{ \dot{u}(x)^2 + x^2 u(x) \} dx$$

Example minimize $F(u)$ in

$$\mathbb{X} := \{ u \in C^1([0,2]) : u(0) = 1, u(2) = 7 \}$$

Observe that $\mathbb{X}$ is an affine space with con. vect. space

$$V := \{ v \in C^1([0,2]) : v(0) = v(2) = 0 \}$$

Assume $u_0 \in \mathbb{X}$ is a min. point, and consider

$$\begin{aligned} \varphi(t) := F(u_0 + tv) &= \int_0^2 \dot{u}_0(x)^2 dx + 2t \int_0^2 \dot{u}_0(x) \dot{v}(x) dx + t^2 \int_0^2 \dot{v}(x)^2 dx \\ &\quad + \int_0^2 x^2 u_0(x) dx + t \int_0^2 x^2 v(x) dx \end{aligned}$$

Then

$$\delta F(u_0, v) = \varphi'(0) = 2 \int_0^2 \dot{u}_0(x) \dot{v}(x) dx + \int_0^2 x^2 v(x) dx$$

Then we integrate by parts...

$$2 \left[\dot{u}_0(x) v(x) \right]_{x=0}^{x=2} - 2 \int_0^2 \ddot{u}_0(x) v(x) dx + \int_0^2 x^2 v(x) dx$$

||
0 because
 $v \in V$

$$\leadsto \int_0^2 \{ -2 \ddot{u}_0(x) + x^2 \} v(x) dx = 0$$

$$\begin{cases} 2\ddot{u}_0(x) = x^2 \rightarrow \text{ELE in diff. form} \\ u_0(0) = 1 \\ u_0(2) = 7 \end{cases} \quad \text{DIRICHLET BC (prescribe the values of } u_0 \text{ at the boundary)}$$

$\leadsto$ there exists a unique solution, which is the unique min. point.

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Example 2 min $\{F(u) : u \in C^1 \dots, \underbrace{u(0) = 3}_{\text{only one BC}}\}$

Now $V = \{v \in C^1([0,2]) : v(0) = 0\}$

As before we obtain that

$$\int_0^2 \{2\ddot{u}_0(x) \dot{v}(x) + x^2 v(x)\} dx = 0 \quad \forall v \in V$$

When we integrate by parts we obtain that

$$2\ddot{u}_0(2)v(2) + \int_0^2 \{-2\ddot{u}_0(x) + x^2\} v(x) dx = 0 \quad \forall v \in V \quad (*)$$

Step 1 If we limit ourselves to $v \in C^1([0,2])$ with $v(0) = v(2) = 0$, or we limit ourselves to $v \in C_c^\infty((0,2))$

we get

$$\int_0^2 \{-2\ddot{u}_0(x) + x^2\} v(x) dx = 0 \quad \forall v \in C_c^\infty((0,2))$$

$\leadsto 2\ddot{u}_0(x) = x^2 \rightarrow \text{AGAIN ELE as before}$

STEP 2 Going back to (*), now we know that

$$\dot{u}_0(2)v(2) = 0$$

$$\forall v \in V$$

It is enough to consider $v \in V$ with $v(2) \neq 0$ to conclude that

$$\dot{u}_0(2) = 0.$$

We have "proved"

$$\begin{cases} 2\ddot{u}_0(x) = x^2 & \rightarrow \text{ELE} \end{cases}$$

$$\begin{cases} u_0(0) = 3 & \rightarrow \text{Dirichlet BC} \end{cases}$$

DBC

$$\begin{cases} \dot{u}_0(2) = 0 & \rightarrow \text{NEUMANN Boundary Condition NBC} \end{cases}$$

Example 3 If we minimize without BCs we obtain

$$\begin{cases} 2\ddot{u}_0(x) = x^2 \\ \dot{u}_0(0) = 0 \\ \dot{u}_0(2) = 0 \end{cases} \quad \text{NBC originated "on the road"}$$

$$\text{Now...} \quad \dot{u}_0(2)v(2) - \dot{u}_0(0)v(0) + \int_0^2 \{-2\ddot{u}_0(x) + x^2\} v(x) dx = 0$$

Step 1 Consider $v \in C_c^\infty((0,2))$ and deduce $2\ddot{u}_0 = x^2$

Step 2 Consider $v \in V = \mathbb{R}$ with $v(2) = 1$ and $v(0) = 0$

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$v(2) = 0$ and $v(0) = 1$

and we deduce the NBC.

Achtung! These problems are NOT Cauchy problems, and there are no existence and uniqueness results

Example 4 minimize with BC $u(0) = u(2) + 7$

The vector space is

$$V = \{v \in C^1([0, 2]) : v(0) = v(2)\}$$

$$\dots \quad \dot{u}_0(2)v(2) - \dot{u}_0(0)v(0) = 0 \quad \forall v \in V$$

We take $v \in V$ s.t. $v(2) = v(0) = 1$ and we obtain that
 $\dot{u}_0(2) = \dot{u}_0(0)$

$$\begin{cases} 2\ddot{u}_0(x) = x^2 \\ u_0(0) = u_0(2) + 7 \\ \dot{u}_0(2) = \dot{u}_0(0) \end{cases}$$

PBC
 $\uparrow$ periodic.

Example 5 Minimize with $\int_0^2 u(x) dx = 9$

$$\text{Now } V = \{v \in C^1([0, 2]) : \int_0^2 v(x) dx = 0\}$$

In the usual way ...

$$\dot{u}_0(2)v(2) - \dot{u}_0(0)v(0) + \int_0^2 \{-2\ddot{u}_0(x) + x^2\}v(x) dx = 0 \quad \forall v \in V$$

Now use $v \in C_c^\infty((0, 2))$ with 0 average and deduce

$$2\ddot{u}_0(x) = x^2 + \lambda$$

$\uparrow$ constant

$\leadsto$ integral = 0 $\leadsto$ (step 2)
NBC

We have "proved" that

$$\left\{ \begin{array}{l} 2\ddot{u}_0(x) = x^2 + \lambda \quad \leftarrow \text{ELE} \\ \dot{u}_0(0) = 0 \\ \dot{u}_0(2) = 0 \end{array} \right\} \text{NBC on the road}$$

$\int_0^2 u_0(x) dx = 9$ given integral constraint

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