

FUNDAMENTAL LEMMA IN THE CALC. OF VARIATIONS

→ FLCV

→ FLCV with 0 average

→ DBR Lemma (Du Bois Reymond)

Given $\mu: \Omega \rightarrow \mathbb{R}$ with $\Omega \subseteq \mathbb{R}^d$ for some $d \geq 1$.

Assume

$$(\star) \quad \int_{\Omega} \mu(x) \varphi(x) dx = 0 \quad \forall \varphi \in C_c^{\infty}(\Omega)$$

$\uparrow$
compact support

FLCV (classical setting)

Assume that $\mu \in C^0(\Omega)$ and $(\star)$

Then $\mu(x) \equiv 0$ for every $x \in \Omega$

FLCV (Lebesgue setting)

Assume that $\mu \in L^1_{loc}(\Omega)$ and $(\star)$.

Then $\mu(x) = 0$ for almost every $x \in \Omega$

$$[\mu \in L^1_{loc}(\Omega) \Leftrightarrow \mu \in L^1(\Omega') \text{ for every } \Omega' \subset\subset \Omega]$$

$\uparrow$
 $\text{clos}(\Omega') \subseteq \Omega$ and compact

FLCV with 0 average (classical setting)

Assume $\mu \in C^0(\Omega)$ and $(\star)$ is true for every $\varphi \in C_c^{\infty}(\Omega)$

such that

$$\int_{\Omega} \varphi(x) dx = 0$$

Then there exists $c \in \mathbb{R}$ s.t. $\mu(x) = c$ for every $x \in \Omega$.

The same is true in the Lebesgue setting with $u \in L^1_{loc}(\Omega)$ and the conclusion is $u(x) = c$ for almost every $x \in \Omega$.

DBR Lemma (Lebesgue setting)

Assume that $(a, b) \subseteq \mathbb{R}$ is an interval, assume that $u \in L^1_{loc}((a, b))$ satisfies

$$\int_a^b u(x) \varphi'(x) dx = 0 \quad \forall \varphi \in C_c^\infty((a, b))$$

Then again $\exists c \in \mathbb{R}$ s.t. $u(x) = c$ for almost every $x \in (a, b)$.

Proof (FLCV - classical) Assume that $u(x) \neq 0$, wlog $\exists x_0 \in \Omega$ s.t. $u(x_0) = \alpha > 0$

Since u is continuous, then

$$\exists r > 0 \text{ s.t. } u(x) \geq \frac{\alpha}{2} \text{ in } B(x_0, r) \subset \subset \Omega$$



There exists $\varphi \in C^\infty(\mathbb{R}^d)$ s.t.

$$(i) \quad 0 \leq \varphi(x) \leq 1 \quad \forall x \in \mathbb{R}^d$$

$$(ii) \quad \varphi(x) = 0 \quad \forall x \text{ in } \mathbb{R}^d \setminus B(x_0, r) \rightsquigarrow \varphi \in C_c^\infty(\Omega)$$

$$(iii) \quad \varphi(x) = 1 \quad \forall x \in B(x_0, \frac{r}{2})$$

$$\int_{\Omega} u(x) \varphi(x) dx = \int_{B(x_0, r)} u(x) \varphi(x) dx$$

$$\geq \int_{B(x_0, \frac{r}{2})} \underbrace{u(x)}_{\geq \frac{\alpha}{2}} \underbrace{\varphi(x)}_1 dx$$

$$= \frac{\alpha}{2} |B(x_0, \frac{r}{2})| > 0.$$

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Proof FLCV (Lebesgue)

Idea: "it would be nice to choose $\varphi(x) = \text{sign } u(x)$ "

Take any $\Omega' \subset \subset \Omega$ (it is enough to consider $\Omega' = \text{any ball}$)

Define $v(x) := \begin{cases} \text{sign}(u(x)) & \text{if } x \in \Omega' \\ 0 & \text{otherwise} \end{cases}$

Then $\exists \{\varphi_n\} \subseteq C_c^\infty(\Omega)$ such that

(i) $\varphi_n(x) \rightarrow v(x)$ for almost every $x \in \Omega$

(ii) $|\varphi_n(x)| \leq 1$ for every $x \in \Omega$ and every $n \in \mathbb{N}$

(iii) support of $\varphi_n \subseteq \Omega'' \subset \subset \Omega$

For every $n \in \mathbb{N}$

$$\int_{\Omega} u(x) \varphi_n(x) dx = 0$$

$\downarrow \leftarrow$ Dominated convergence

$$\int_{\Omega} u(x) v(x) dx = \int_{\Omega'} |u(x)| dx \leadsto u(x) \equiv 0 \text{ for almost every } x \in \Omega'$$

$$|u(x) \varphi_n(x)| \leq |u(x) \cdot \underset{\substack{\uparrow \\ \text{characteristic function of } \Omega''}}{1_{\Omega''}}(x)|$$

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FLCV with 0 average.

The key idea is the following

$$\int_{\Omega} u(x) \varphi(x) dx = 0 \quad \forall \varphi \in C_c^\infty(\Omega) \text{ and } \int_{\Omega} \varphi(x) dx = 0$$

then for every $c \in \mathbb{R}$ it turns out that

$$\int_{\Omega} (u(x) - c) \varphi(x) dx = 0 \quad \forall \varphi \text{ as above}$$

Idea: choose c s.t. $u(x) - c$ has 0 average, and then use $\varphi = u - c$ or $\varphi = \text{sign}(u - c)$

let us take any $\Omega' \subset \subset \Omega$ (balls are enough). let

$$c := \frac{1}{|\Omega'|} \int_{\Omega'} u(x) dx$$

let us consider $\text{sign}(u(x) - c)$ and let us choose $d \in \mathbb{R}$ s.t.

$$\int_{\Omega'} (\text{sign}(u - c) - d) dx = 0$$

Approximation result: $\exists \{\varphi_m\} \subseteq C_c^\infty(\Omega)$ s.t.

$$(i) \quad \varphi_m(x) \rightarrow \begin{cases} \text{sign}(u - c) - d & \text{in } \Omega' \\ 0 & \text{otherwise} \end{cases}$$

$$(ii) \quad \exists M \in \mathbb{R} \text{ s.t. } |\varphi_m(x)| \leq M \quad \forall x \in \Omega \quad \forall m \in \mathbb{N}$$

$$(iii) \quad \text{support of } \varphi_m \subseteq \Omega'' \subset \subset \Omega.$$

$$(iv) \quad \int_{\Omega} \varphi_m(x) dx = 0$$

For every $m \in \mathbb{N}$

$$\int_{\Omega} (u(x) - c) \varphi_m(x) dx = 0$$

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$$\int_{\Omega'} (u(x) - c) [\text{sign}(u(x) - c) - d] dx$$

$$= \int_{\Omega'} \{|u(x) - c| - d(u(x) - c)\} dx = \int_{\Omega'} |u(x) - c| dx$$

This proves that $u(x) \equiv c$ for almost every $x \in \Omega'$.

In order to conclude, we need to show that c does not depend on Ω' .

Proof of DBR It is enough to observe that

$$\{\varphi'(x) : \varphi \in C_c^\infty((a,b))\} = \{\psi(x) : \psi \in C_c^\infty \text{ and } \int_a^b \psi dx = 0\}$$

$$\int_a^b \varphi'(x) dx = \varphi(b) - \varphi(a) = 0$$

Remark In DBR we can replace (a,b) by $(a, +\infty)$ or $\mathbb{R}$, but connectedness is essential.

Ex Give a proof of FLCV with 0 average in the classical setting.

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How to approx with 0 average.

Assume $v(x)$ has 0 average and $v_m \in C_c^\infty$ is s.t.

$v_m \rightarrow v(x)$ (pointwise + domination)

$$\text{In any case } \int_{\Omega} v_m(x) dx \xrightarrow{C_m} \int_{\Omega} v(x) dx = 0$$

Consider $\hat{v}_m(x) := v_m(x) - c_m \psi(x)$

where ψ is any function of class C_c^∞ with $\int_{\Omega} \psi(x) dx = 1$