

Integral functionals

SYNONIMS but in different contexts

- Function
- Functional
- Map (mapping)
- Application
- Operator

Let $(a, b) \subseteq \mathbb{R}$ be an interval, and let

$L: (a, b) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a function
 $\uparrow$ LAGRANGIAN (with some regularity)

Then we can consider

$$F: \mathbb{X} \rightarrow \mathbb{R}$$

where $\mathbb{X} \subseteq C^1([a, b]) = \{ \text{functions } u: [a, b] \rightarrow \mathbb{R} \text{ of class } C^1 \}$

defined by

$$F(u) := \int_a^b L(x, u(x), \dot{u}(x)) dx$$

We can extend the notion in several ways

- by allowing $u: [a, b] \rightarrow \mathbb{R}^m$
- " " $u: \Omega \rightarrow \mathbb{R}$ with $\Omega \subseteq \mathbb{R}^d$
- " " higher order derivatives
- " " $(a, +\infty)$ instead of (a, b) .

$$L(x, s, p)$$

$\uparrow$
position

$\nwarrow$
derivative
velocity

Classical examples

$$F(u) = \int_a^b \dot{u}(x)^2 dx \quad G(u) = \int_a^b |\dot{u}(x)| dx \quad H(u) = \int_a^b |\dot{u}(x)|^{1/2} dx$$

Compute $\min \{ F(u) : \underbrace{u \in C^1([0,1])}_{\text{set } \mathbb{X}}, \underbrace{u(0)=1, u(1)=0}_{\text{Boundary conditions}} \}$

INDIRECT METHOD Observe that $\mathbb{X}$ is an affine space with corresponding vector space

$$V = \{ v \in C^1([0,1]) : v(0) = v(1) = 0 \}$$

\begin { brutal }

Assume $u_0(x)$ is a min. point. in $\mathbb{X}$. For every $v \in V$ we can define

$$\begin{aligned} F(u_0 + tv) &= \int_0^1 (\dot{u}_0(x) + t \dot{v}(x))^2 dx \\ &= \int_0^1 \dot{u}_0(x)^2 dx + 2t \int_0^1 \dot{u}_0(x) \dot{v}(x) dx + t^2 \int_0^1 \dot{v}(x)^2 dx \\ &=: \varphi(t) \quad (t \in \mathbb{R}) \end{aligned}$$

$$\delta F(u_0, v) = \varphi'(0) = 2 \int_0^1 \dot{u}_0(x) \dot{v}(x) dx$$

If $u_0(x) \in \mathbb{X}$ is a min. point. then

$$\int_0^1 \dot{u}_0(x) \dot{v}(x) dx = 0 \quad \forall v \in V$$

First integral
form of
ELE

Euler-Lagrange equ.

Now I integrate by parts:

$$\int_0^1 \ddot{u}_0(x) \dot{v}(x) dx = \left[\underbrace{\ddot{u}_0(x)}_{\parallel} v(x) \right]_{x=0}^{x=1} - \int_0^1 \ddot{u}_0(x) v(x) dx$$
$$\underbrace{\ddot{u}_0(1)}_0 v(1) - \underbrace{\ddot{u}_0(0)}_0 v(0)$$

Therefore

$$\boxed{\int_0^1 \ddot{u}_0(x) v(x) dx = 0 \quad \forall v \in V} \quad \text{Second integral form of ELE}$$

Achtung! I can not integrate by parts because I do not know that $\ddot{u}_0$ exists, but I DO NOT CARE.

It is reasonable to deduce that $\ddot{u}_0(x) = 0$ for every $x \in [0, 1]$

$$\begin{cases} \ddot{u}_0(x) = 0 & \leftarrow \text{ELE in differential form} \\ u_0(0) = 1 \\ u_0(1) = 9 \end{cases} \quad \text{BCs (boundary conditions)}$$

The unique solution is $u_0(x) = 1 + 8x$

I conjecture this is the unique min point.

end {brutal}

Let us prove that $u_0(x)$ is the unique min. point.

Any other competitor $u(x) \in \mathbb{X}$ can be written as

$$u(x) = u_0(x) + \underbrace{v(x)}_{\downarrow \in V}$$

= 0 because now I can
↗ integrate by parts

$$F(u) = \int_0^1 (\dot{u}_0 + \dot{v})^2 dx = \int_0^1 \dot{u}_0^2 dx + 2 \int_0^1 \dot{u}_0 \dot{v} dx + \int_0^1 \dot{v}^2 dx = 0$$
$$\geq F(u_0)$$

with equality if and only if $\int_0^1 \dot{v}^2 dx = 0$, namely

iff $\dot{v}(x) \equiv 0 \leadsto v(x)$ is constant $\leadsto v(x) \equiv 0$ because $v(0) = 0$.

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min $\left\{ \int_0^1 |\dot{u}(x)| dx : u \in C^1([0,1]), u(0)=1, u(1)=9 \right\}$

$$\int_0^1 |\dot{u}(x)| dx \geq \int_0^1 \dot{u}(x) dx = u(1) - u(0) = 8$$

↑
equality iff $\dot{u}(x) \geq 0$ for every $x \in [0,1]$

In this case the min exists, BUT there are infinitely many minimizers (any $u_0(x)$ with $\dot{u}_0(x) \geq 0$ and $u_0(0)=1$ and $u_0(1)=9$ is a minimizer).

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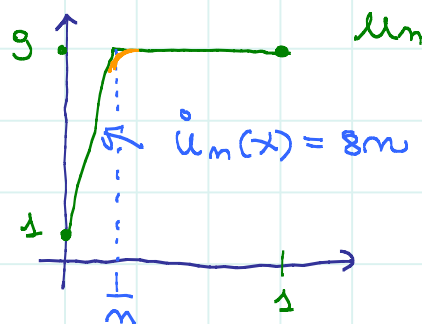
min $\left\{ \int_0^1 |\dot{u}(x)|^{1/2} dx : u \in C^1(\dots), u(0)=1, u(1)=9 \right\}$

Surprise! The min does NOT exist, and $\inf = 0$.

Easy: if $\inf = 0$, the min does NOT exist (it should be $\dot{u}(x) \equiv 0 \leadsto u(x)$ constant $\leadsto$ NO BCs)

Enough to FIND a sequence $\{u_m\} \subseteq \mathbb{X}$ s.t. $F(u_m) \rightarrow 0$

$$F(u_m) = \int_0^1 \sqrt{8m} dx = \frac{1}{m} \cdot \sqrt{8m} \rightarrow 0$$



We can modify u_m around $\frac{1}{m}$ in such a way that it is of class C^1

The difference between the three examples is the convexity of $L(x, s, p)$ with respect to p

$L(x, s, p) = p^2 \rightsquigarrow$ existence and uniqueness

$L(x, s, p) = |p| \rightsquigarrow$ existence without uniqueness

$L(x, s, p) = \sqrt{|p|} \rightsquigarrow$ big troubles

The ambient space is important.

For standard Lagrangians the infimum in C^1 coincides with the infimum in piecewise- C^1 .

It is enough to make a smooth connection near the intermediate points.

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