

Introduction to Calculus of Variations

General problem Set X and a function $F: X \rightarrow \mathbb{R}$

sometimes

$$\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$$

Goal: determine whether there exists

$$\min \{ F(x) : x \in X \}$$

← number (unique if it exists)

→ If min exists, then find

$$\operatorname{argmin} \{ F(x) : x \in X \}$$

← set of min. points (a subset of X , with one or more elements)

$$= \{ x \in X : F(x) = \min \{ F(x) : x \in X \} \}$$

→ If min does NOT exist, then find inf & characterize minimizing sequences, namely $\{x_n\} \subset X$ s.t.

$$F(x_n) \rightarrow \inf$$

"inf-izing sequence"

Two main approaches

- INDIRECT METHOD
- DIRECT METHOD

INDIRECT METHOD Road map

- ① Trying to figure out which are the min. points
- ② Proving they are min. points.

Example $X: \mathbb{R}^2$ $f(x,y) := x^2 + y^2 + 2x - 6y$

① $\nabla f = (2x+2, 2y-6) = (0,0) \Leftrightarrow x=-1, y=3$

I conjecture that $(x_0, y_0) = (-1, 3)$ is the unique min. point

② I prove that (x_0, y_0) is the min. point.

Every other competitor can be written as

$$(x, y) = (-1+u, 3+v) = (x_0+u, y_0+v)$$

$$\begin{aligned} f(x, y) &= (x_0+u)^2 + (y_0+v)^2 + 2(x_0+u) - 6(y_0+v) \\ &= \underbrace{x_0^2}_{\text{orange}} + \underbrace{2x_0u}_{\text{blue}} + \underbrace{u^2}_{\text{green}} + \underbrace{y_0^2}_{\text{orange}} + \underbrace{2y_0v}_{\text{blue}} + \underbrace{v^2}_{\text{green}} + \underbrace{2x_0}_{\text{orange}} + \underbrace{2u}_{\text{blue}} - \underbrace{6y_0}_{\text{orange}} - \underbrace{6v}_{\text{blue}} \\ &= f(x_0, y_0) + \underbrace{(2x_0+2)u + (2y_0-6)v}_{\substack{0 \\ \text{by definition of } (x_0, y_0)}} + (u^2+v^2) \\ &\geq f(x_0, y_0) \quad \text{with } = \text{ iff } u=v=0. \end{aligned}$$

DIRECT METHOD Road map

- ① Proving that min. exists
- ② Trying to find min point

In the same example we can observe that

$$\lim_{x^2+y^2 \rightarrow +\infty} f(x,y) = +\infty$$

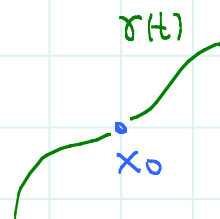
Thanks to generalized Weierstrass theorem we know that min exists (we can restrict to $\{f(x,y) \leq 2021\}$)

Now we proceed as in step 1 before.

INDIRECT METHOD IN GENERAL

Let X be a set, and let $F: X \rightarrow \mathbb{R}$ be a function

Let $x_0 \in X$ be a point, let $\delta > 0$, let $\gamma: (-\delta, \delta) \rightarrow X$ be a curve with $\gamma(0) = x_0$



I can consider $\varphi: (-\delta, \delta) \rightarrow \mathbb{R}$ defined by

$$\varphi(t) := F(\gamma(t))$$

I can compute $\varphi'(0)$. If it exists, I call it the first variation of F in x_0 wrt the curve γ

$$\delta F(x_0, \gamma) := \varphi'(0)$$

Theorem (Necessary condition) If x_0 is a min point for F , then

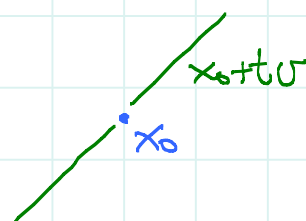
$$\delta F(x_0, \gamma) = 0$$

whenever it exists.

Equivalent statement: for every curve with $\gamma(0) = x_0$, we have that $\varphi'(0)$

→ either it does not exist

→ or is equal to 0.



Special case X is an AFFINE space with corresponding vector space V .

For every $x_0 \in X$ and $v \in V$ we can consider

$$\varphi(t) := F(x_0 + tv)$$

In this case

$$\varphi'(0) = \delta F(x_0, v)$$

GATEAUX derivative of F
in x_0 in the direction v

↑ usual directional derivative

DIRECT METHOD IN GENERAL

Def. Let X be a set.

Let $\text{Seq}(X)$ be the set of sequences with values in X

A NOTION OF CONVERGENCE in X is any subset of
 $\text{Seq}(X) \times X$

In brutal words, it is the list of converging sequences in X
with their limits.

WITHOUT ANY RULE AT ALL.

Def. We say that X is compact if every sequence admits
a converging subsequence (a subseq. in the list)

We say that a subset $K \subseteq X$ is compact if for
every sequence $\{x_n\} \subseteq K$ there exist $n_k \rightarrow +\infty$ and
 $x_\infty \in K$ such that $x_{n_k} \rightarrow x_\infty$

Def. A function $\varphi : X \rightarrow \mathbb{R}$ is lower semicontinuous (LSC)
if for every converging sequence $x_n \rightarrow x_\infty$ it turns out
that

$$\liminf_{n \rightarrow \infty} \varphi(x_n) \geq \varphi(x_\infty)$$

Theorem (Weierstrass) Let $F: X \rightarrow \mathbb{R}$. Assume that

(i) X is compact and $\neq \emptyset$

(ii) F is LSC

Then $\min \{ F(x) : x \in X \}$ exists.

Proof. Let $I := \inf \{ F(x) : x \in X \} \in \mathbb{R} \cup \{-\infty\}$

Let $\{x_n\} \subseteq X$ be a inf-izing sequence.

Since X is compact, there exists $x_{n_k} \rightarrow x_\infty$. Now

$$\begin{array}{ccccccc} I \leq F(x_\infty) \leq \liminf_{k \rightarrow \infty} F(x_{n_k}) = \lim_{n \rightarrow \infty} F(x_{n_k}) = I. \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \text{defn. of} & F \text{ is LSC} & \text{subsequence} & \text{of conv. sequence} & \text{defn. of } x_n \\ \text{inf} & & \text{of numbers.} & & \end{array}$$

This proves that $F(x_\infty) = I \Rightarrow x_\infty$ is a min. point and $I \in \mathbb{R}$.
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Def. A function $F: X \rightarrow \mathbb{R}$ is called COERCIVE if there exists a compact set $K \subseteq X$ s.t.

$$\inf \{ F(x) : x \in X \} = \inf \{ F(x) : x \in K \}$$

Theorem (Generalized W) Assume that

(i) F is coercive

(ii) F is LSC.

Then again min exists.

Proof: exercise

Remark F is coercive provided that there exists $M \in \mathbb{R}$ such that

$\{x \in X : F(x) \leq M\}$ is $\neq \emptyset$ and compact (or contained in a compact set).