

## GAGLIARDO - BRASCAMP - LIEB

**Lemma** (Gagliardo 1958)

$\downarrow \downarrow$   
 Sia  $d \geq 2$  un intero. Per ogni  $i \in \{1, \dots, d\}$  sia

$$\varphi_i : \mathbb{R}^{d-1} \rightarrow [0, +\infty) \text{ con } \varphi_i \in L^{d-1}(\mathbb{R}^{d-1}) \cap C_0^\infty(\mathbb{R}^{d-1})$$

$$P_i : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1} \text{ la proiezione che dimentica la coord. } i\text{-esima}$$

Consideriamo la funzione  $\varphi : \mathbb{R}^d \rightarrow [0, +\infty)$  definita da

$$\varphi(x) := \prod_{i=1}^d \varphi_i(P_i x) \quad \forall x \in \mathbb{R}^d$$

Allora  $\varphi \in L^1(\mathbb{R}^d)$  e vale

$$\|\varphi\|_{L^1(\mathbb{R}^d)} \leq \prod_{i=1}^d \|\varphi_i\|_{L^{d-1}(\mathbb{R}^{d-1})}$$

**Dim** Inclusione su  $d$ . **$d=2$**  Quasi banale  $\varphi(x, y) = \varphi_1(y) \cdot \varphi_2(x)$ 

$$\int_{\mathbb{R}^2} \varphi(x, y) dx dy = \int_{\mathbb{R}} \varphi_2(x) dx \int_{\mathbb{R}} \varphi_1(y) dy$$

$$\|\varphi\|_{L^1(\mathbb{R}^2)} = \|\varphi_2\|_{L^1(\mathbb{R})} \cdot \|\varphi_1\|_{L^1(\mathbb{R})}$$

Per capire come funziona, vediamo il passaggio da  $d=3$  a  $d=4$ .

$$\varphi(x, y, z, w) = a(y, z, w) b(x, z, w) c(x, y, w) d(x, y, z)$$

$$\int_{\mathbb{R}^4} \varphi(x, y, z, w) dx dy dz dw =$$

$$= \int_{\mathbb{R}^3} d(x, y, z) dx dy dz \underbrace{\int_{\mathbb{R}} a(y, z, w) b(x, z, w) c(x, y, w) dw}_{\Phi(x, y, z)}$$

$$= \int_{\mathbb{R}^3} d(x, y, z) \underbrace{\Phi(x, y, z)}_{\frac{2}{3}} dx dy dz$$

$$\leq \underbrace{\left\{ \int_{\mathbb{R}^3} d(x, y, z)^3 dx dy dz \right\}^{1/3}}_{\|d\|_{L^3(\mathbb{R}^3)}} \cdot \underbrace{\left\{ \int_{\mathbb{R}^3} \Phi(x, y, z)^{3/2} dx dy dz \right\}^{2/3}}_{\text{resto}}$$

Quindi ci basta dim. che  $\text{resto} \leq \|a\|_{L^3(\mathbb{R}^3)} \cdot \|b\| \dots \cdot \|c\| \dots$   
Ora

$$\Phi(x, y, z) = \int_{\mathbb{R}} \underbrace{a(y, z, w)}_{\frac{1}{3}} \underbrace{b(x, z, w)}_{\frac{1}{3}} \underbrace{c(x, y, w)}_{\frac{1}{3}} dw$$

$$\leq \left\{ \int_{\mathbb{R}} a(y, z, w)^3 dw \right\}^{1/3} \left\{ \int_{\mathbb{R}} b(x, z, w)^3 dw \right\}^{1/3} \left\{ \int_{\mathbb{R}} c \dots \right\}^{1/3}$$

$$\Phi(x, y, z)^{3/2} \leq \underbrace{\left\{ \int_{\mathbb{R}} a(y, z, w)^3 dw \right\}^{1/2}}_{A(y, z)} \underbrace{\left\{ \int_{\mathbb{R}} b(x, z, w)^3 dw \right\}^{1/2}}_{B(x, z)} \underbrace{\left\{ \dots \right\}^{1/2}}_{C(x, y)}$$

$$\int_{\mathbb{R}^3} \Phi(x, y, z) dx dy dz \leq \int_{\mathbb{R}^3} A(y, z) B(x, z) C(x, y) dx dy dz$$

Hp  
involuttiva

$$\leq \left\{ \int_{\mathbb{R}^2} A(y, z)^2 dy dz \right\}^{1/2} \left\{ \int_{\mathbb{R}^2} B(x, z)^2 dx dz \right\}^{1/2} \dots \left\{ \dots \right\}^{1/2}$$

Ora

$$\int_{\mathbb{R}^2} A(y, z)^2 dy dz = \int_{\mathbb{R}^2} dy dz \int_{\mathbb{R}} a(y, z, w)^3 dw$$

$$= \int_{\mathbb{R}^3} a(y, z, w)^3 dy dz dw$$

$$= \|a\|_{L^3(\mathbb{R}^3)}^3$$

Questi integrali li elevo alla  $\frac{1}{2}$  !

$$\int_{\mathbb{R}^3} \Phi(x, y, z) dx dy dz \leq \|a\|_{L^3(\mathbb{R}^3)}^{3/2} \|b\|^{3/2} \dots \|c\|^{3/2}$$

Ora non resta che elevare alla  $\frac{2}{3}$ .

Consiglio: provare a fare il passaggio da  $d=2$  a  $d=3$

Passo vero da  $d$  a  $d+1$

Notazione:  $\hat{x}_i = x$  senza componente  $i$

$\hat{x}_{i, d+1} = x$  senza componenti  $i$  e  $d+1$

$$\varphi(x) = \varphi_1(\hat{x}_1) \cdots \varphi_d(\hat{x}_d) \cdot \varphi_{d+1}(\hat{x}_{d+1})$$

$$\int_{\mathbb{R}^{d+1}} \varphi(x) dx = \int_{\mathbb{R}^d} \varphi_{d+1}(\hat{x}_{d+1}) d\hat{x}_{d+1} \underbrace{\int_{\mathbb{R}} \varphi_1(\hat{x}_1) \cdots \varphi_d(\hat{x}_d) dx_{d+1}}_{\Phi(\hat{x}_{d+1})}$$

$$= \int_{\mathbb{R}^d} \varphi_{d+1}(\hat{x}_{d+1}) \cdot \underbrace{\Phi(\hat{x}_{d+1})}_{\frac{d-1}{d}} \cdot d\hat{x}_{d+1}$$

$$\leq \underbrace{\left\{ \int_{\mathbb{R}^d} \varphi_{d+1}(\hat{x}_{d+1})^d d\hat{x}_{d+1} \right\}^{1/d}}_{\|\varphi_{d+1}\|_{L^d(\mathbb{R}^d)}} \underbrace{\left\{ \int_{\mathbb{R}^d} \Phi(\dots)^{\frac{d}{d-1}} \right\}^{\frac{d-1}{d}}}_{\text{resto}}$$

e spero che  $\text{resto} \leq \|\varphi_1\|_{L^d(\mathbb{R}^d)} \cdots \|\varphi_d\|_{L^d(\mathbb{R}^d)}$

$$\Phi(\hat{x}_{d+1}) = \int_{\mathbb{R}} \underbrace{\varphi_1(\hat{x}_1)}_{\frac{1}{d}} \cdots \underbrace{\varphi_d(\hat{x}_d)}_{\frac{1}{d}} d\hat{x}_{d+1}$$

$$\leq \left\{ \int_{\mathbb{R}} \varphi_1(\hat{x}_1)^d d\hat{x}_{d+1} \right\}^{\frac{1}{d}} \cdots \left\{ \int_{\mathbb{R}} \varphi_d(\hat{x}_d)^d d\hat{x}_{d+1} \right\}^{1/d}$$

$$\Phi(\hat{x}_{d+1})^{\frac{d}{d-1}} \leq \underbrace{\left\{ \int_{\mathbb{R}} \varphi_1(\hat{x}_1)^d d\hat{x}_{d+1} \right\}^{\frac{1}{d-1}}}_{\Phi_1(\hat{x}_{1,d+1})} \cdots \underbrace{\left\{ \int_{\mathbb{R}} \varphi_d(\hat{x}_d)^d d\hat{x}_{d+1} \right\}^{\frac{1}{d-1}}}_{\Phi_d(\hat{x}_{d,d+1}) \cdots}$$

$$\int_{\mathbb{R}^d} \Phi(\hat{x}_{d+1})^{\frac{d}{d-1}} \leq \int_{\mathbb{R}^d} \Phi_1(\hat{x}_{1,d+1}) \cdots \Phi_d(\hat{x}_{d,d+1}) d\hat{x}_{d+1}$$

ipotesi  
induttiva  $\leq \left\{ \int_{\mathbb{R}^{d-1}} \Phi_1(\hat{x}_{1,d+1})^{d-1} d\hat{x}_{1,d+1} \right\}^{\frac{1}{d-1}} \cdots$

Chi sono i termini del prodotto al RHS?

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} \bar{\Phi}_1(\hat{x}_{1,d+1}) d\hat{x}_{1,d+1} &= \int_{\mathbb{R}^{d-1}} d\hat{x}_{1,d+1} \int_{\mathbb{R}} \varphi_1(\hat{x}_1)^d dx_{d+1} \\ &= \int_{\mathbb{R}^d} \varphi_1(\hat{x}_1)^d d\hat{x}_1 \end{aligned}$$

Fortunatamente gli esponenti si sistemano quando alla fine elevo alla  $\frac{d-1}{d}$ .

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Esempio  $H^1$  in dimensione 2  $W^{1,2}$

Notazione: dopo aver dimostrato  $H = W$ , per i Sobolev si usa sempre la notazione  $W^{m,p}$ .  
 $W^{m,2}$  si abbrevia di solito con  $H^m$

$H^1(\Omega)$  con  $\Omega = \text{cerchio}$  oppure  $\Omega = \mathbb{R}^2$

Esempio di funzione  $u \in H^1(\Omega)$  ma non  $L^\infty(\Omega)$

Funzioni radiali  $u(x) = \psi(|x|)$

Analisi 2:  $\nabla u(x) = \psi'(|x|) \cdot \frac{x}{|x|}$

Primo tentativo  $\psi(r) = \frac{1}{r^a}$

In quali  $L^p$  sta? In quali  $W^{1,p}$  sta?

$$\int_0^1 \frac{1}{r^{ap}} \cdot r dr \in L^p$$

$$\int_0^1 \frac{1}{r^{(a+1)p}} r dr \in W^{1,p}$$

$$\in L^p \Leftrightarrow ap-1 < 1 \quad p < \frac{2}{a}$$

$$\in W^{1,p} \Leftrightarrow (a+1)p-1 < 1 \quad p < \frac{2}{a+1}$$

Se voglio stare in  $W^{1,2}$  devo prendere "a=0" e ancora non va bene

Si vede che  $|\log r| \notin W^{1,2}(\text{disco})$  per pochissimo

$$|\log |\log r|| \in W^{1,2}(\text{disco}) \quad \text{😊}$$

Stesso esempio funziona in dim. d con  $W^{1,d}$

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