

$$\ddot{u} + 2\delta A^\sigma \dot{u} + c(t) Au = 0$$

Subcritical case: $\sigma \in [0, \frac{1}{2}]$

Statement Assume strict hyperb $0 < \mu_1 \leq c(t) \leq \mu_2$
 Assume $c(t)$ is ω -continuous
 Assume $\sigma \leq \frac{1}{2}$
 Assume

$$4\delta^2 \mu_1 \geq L^2 + 2\delta L \quad (*)$$

where

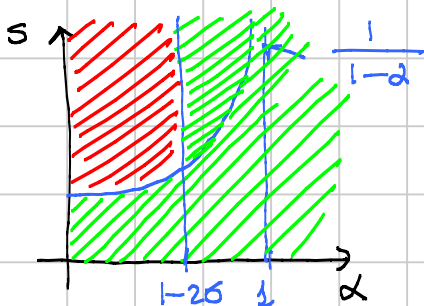
$$L := \limsup_{\varepsilon \rightarrow 0^+} \frac{\omega(\varepsilon)}{\varepsilon^{1-2\sigma}}$$

Then the problem is well posed in $D(A^{1/2}) \times H$ (gap = $\frac{1}{2}$)

Special case Assume $\omega(\varepsilon) = M \varepsilon^\alpha$ for some $\alpha \in (0, 1)$

$$L := \lim_{\varepsilon \rightarrow 0^+} \frac{M \varepsilon^\alpha}{\varepsilon^{1-2\sigma}}$$

- If $\alpha > 1-2\sigma$, then $L = 0$, and $(*)$ is satisfied
 $\sigma > \frac{1-\alpha}{2}$
- If $\alpha = 1-2\sigma$, then $L = M$ has to be small enough
- If $\alpha < 1-2\sigma$, then $L = +\infty$ and $(*)$ is not satisfied
 In that case there are counterexamples (see lecture 8)



Heuristics → Effect of dissipation

$$E(t) \leq E(0) \exp(-4\delta \lambda^{2\sigma} t)$$

→ Effect of low regularity (see Lecture 2)

$$E(t) \leq E(0) \exp\left(\lambda \omega\left(\frac{1}{\lambda}\right) t\right)$$

→ Superposition

$$E(t) \leq E(0) \exp\left(-4\delta \lambda^{2\sigma} t + \lambda \omega\left(\frac{1}{\lambda}\right) t\right)$$

$$\text{Negative wins} \Leftrightarrow 4\delta \lambda^{2\sigma} >> \lambda \omega\left(\frac{1}{\lambda}\right)$$

$$\Leftrightarrow \lambda^{1-2\sigma} \omega\left(\frac{1}{\lambda}\right) \text{ is small for large } \lambda$$

$$\Leftrightarrow \frac{\omega(\varepsilon)}{\varepsilon^{1-2\sigma}} \ll 1 \quad \text{for } \varepsilon \ll 1.$$

$$\lambda = \frac{1}{\varepsilon}$$

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MAIN TOOL σ correction of hyperb. energy

$$F(t) := \underbrace{(\dot{u} + \delta \lambda^{2\sigma} u)^2 + \delta^2 \lambda^{4\sigma} u^2}_{\text{kov. energy}} + \lambda^2 \gamma(t) u^2$$

↑
has to be choosed
(will be $\gamma(t) = c_\varepsilon(t)$ for
some ε)

Hope: compute $F_k(t)$ (= energy of k -th component)
and prove that

$$F_k'(t) \leq 0$$

Proof.

$$\begin{aligned}
 F'(t) &= -2\delta \lambda^{2\sigma} \dot{u}^2 - 2\delta \lambda^{2+2\sigma} c(t) u^2 - 2\lambda^2 c(t) u \dot{u} \\
 &\quad + \lambda^2 \dot{\sigma}(t) u^2 + 2\lambda^2 \sigma(t) u \dot{u} \\
 &= \underbrace{-2\delta \lambda^{2\sigma} \dot{u}^2}_A - 2 \underbrace{\left(\delta \lambda^{2+2\sigma} c(t) - \frac{1}{2} \lambda^2 \dot{\sigma}(t) \right)}_B u^2 \\
 &\quad - 2 \underbrace{\lambda^2 (c(t) - \sigma(t))}_C u \dot{u}
 \end{aligned}$$

As before we have a quadratic form in $u \dot{u}$

$$4AB \geq C^2$$

$$4\delta \lambda^{2\sigma} \left(\delta \lambda^{2+2\sigma} c(t) - \frac{1}{2} \lambda^2 \dot{\sigma}(t) \right) \geq \lambda^4 (c(t) - \sigma(t))^2$$

Choose $\sigma(t) := c_\varepsilon(t) = \frac{1}{\varepsilon} \int_t^{t+\varepsilon} c(s) ds$

$$(i) \quad |\dot{c}_\varepsilon(t)| \leq \frac{\omega(\varepsilon)}{\varepsilon} \quad (ii) \quad |c(t) - c_\varepsilon(t)| \leq \omega(\varepsilon)$$

It is enough to prove that

$$4\delta \lambda^{2\sigma} \left(\delta \lambda^{2+2\sigma} \underset{\substack{\uparrow \\ \text{strict} \\ \text{hyperb.}}}{\mu_1} - \frac{1}{2} \lambda^2 \frac{\omega(\varepsilon)}{\varepsilon} \right) \geq \lambda^4 \omega(\varepsilon)^2$$

Take $\varepsilon = \frac{1}{\lambda}$ and obtain

$$4\delta^2 \mu_1 \lambda^{2+4\sigma} \geq 2\delta \lambda^{2\sigma+3} \omega\left(\frac{1}{\lambda}\right) + \lambda^4 \omega\left(\frac{1}{\lambda}\right)^2$$

$$4\delta^2 \mu_1 \geq \underbrace{2\delta \lambda^{1-2\sigma} \omega\left(\frac{1}{\lambda}\right)}_{L} + \underbrace{\lambda^{2-4\sigma} \omega\left(\frac{1}{\lambda}\right)^2}_{L^2}$$

This is condition (*)

Therefore

$$F_k'(t) \leq 0$$

for k large (large frequencies)

$$\Rightarrow F_k(t) \leq F_k(0)$$

$$\Rightarrow \lambda_k^2 u_k^2(t) \leq M_1 F_k(0) \leq M_2 (u_{1k}^2 + \lambda_k^2 u_{0k}^2)$$

$$\dot{u}_k^2(t) \leq M_3 F_k(0) \leq M_4 (u_{1k}^2 + \lambda_k^2 u_{0k}^2)$$

As before, this leads to $(u(t), \dot{u}(t)) \in D(A^{1/2}) \times H$

$$\underbrace{(u_0, u_1)}_{\substack{\longrightarrow \quad \longrightarrow \quad \longrightarrow}} \in D(A^{\alpha+\frac{1}{2}}) \times D(A^\alpha) \Rightarrow (u(t), \dot{u}(t)) \in \text{same space.}$$

Remark 1 Again boundedness of high frequencies

Remark 2 Again Gevrey regularity with $\varphi(\lambda) = \lambda^{2\sigma}$

Need to prove that

$$F_k'(t) \leq -\underset{\uparrow}{c} \lambda_k^{2\sigma} F_k(t)$$

$c = c(t)$ due to strict hyperbolicity