

$$\ddot{u} + 2\delta A^\sigma \dot{u} + c(t) Au = 0$$

Main issue: competition between damping and oscillations of $c(t)$

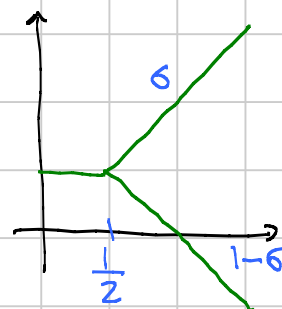
Supercritical case: either $\sigma > \frac{1}{2}$
or $\sigma = \frac{1}{2}$ and $\delta \gg 1$

Main result Assume supercrit. case
Assume $0 \leq c(t) \leq \mu_2$
 $\uparrow$ weak hypob
 $\uparrow$ boundedness

No further regularity of $c(t)$

Then dissipation wins, so that we have well posedness in

$$D(A^\alpha) \times D(A^\beta) \quad \text{with} \quad 1 - \sigma \leq \alpha - \beta \leq \sigma$$



Heuristics

→ Assume no dissipation at all. Then

$$E(t) \leq E(0) \exp \left\{ \lambda \int_0^t (1 + |c(s)|) ds \right\} \quad (\text{kov. estimate})$$

$$\leq E(0) \exp \left\{ \lambda (t + \mu_2 t) \right\}$$

(effect of oscillations)

→ Assume $c(t)$ is constant. Then for every component

$$E'(t) \sim -4\delta \lambda^{2\sigma} |\dot{u}|^2 \sim -4\delta \lambda^{2\sigma} E(t)$$

$$E(t) \leq E(0) \exp\{-4\delta \lambda^{2\sigma} t\}$$

(effect of dissipation)

→ Assume we have oscillations and dissipation.

Assume we can add the contributions:

$$E(t) \leq E(0) \exp\left\{ \underbrace{-4\delta \lambda^{2\sigma} t}_{\text{dissip}} + \underbrace{\lambda t + \mu_2 \lambda t}_{\text{low reg. of } c(t)} \right\}$$

Minus sign wins : • if $\sigma > \frac{1}{2}$
• if $\sigma = \frac{1}{2}$ and δ is large enough.
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Main tool : σ -connection of usual Kowalestian energy

$$\begin{aligned} E(t) &:= (\dot{u} + \delta \lambda^{2\sigma} u)^2 + \delta^2 \lambda^{4\sigma} u^2 \\ &= \dot{u}^2 + 2\delta^2 \lambda^{4\sigma} u^2 + 2\delta \lambda^{2\sigma} u \dot{u} \end{aligned}$$

Proof. Let us consider components $u_k(t) := \langle u(t), e_k \rangle$
which are solutions to

$$\ddot{u}_k + 2\delta \lambda_k^{2\sigma} \dot{u}_k + c(t) \lambda_k^2 u_k = 0$$

and considere $E_k(t) :=$ energy of u_k defined as above.

$$\begin{aligned}
E' &= 2\dot{u}\ddot{u} + 4\delta^2\lambda^{4\delta}u\dot{u} + 2\delta\lambda^{2\delta}\dot{u}^2 + 2\delta\lambda^{2\delta}u\ddot{u} \\
&= 2\dot{u}(-2\delta\lambda^{2\delta}\dot{u} - \lambda^2 c(t)u) + \cancel{4\delta^2\lambda^{4\delta}u\dot{u}} + \cancel{2\delta\lambda^{2\delta}\dot{u}^2} \\
&\quad + 2\delta\lambda^{2\delta}u(-\cancel{2\delta\lambda^{2\delta}\dot{u}} - \lambda^2 c(t)u) \\
&= -2\delta\lambda^{2\delta}\dot{u}^2 - 2\delta\lambda^{2+2\delta}c(t)u^2 - 2\lambda^2 c(t)u\dot{u}
\end{aligned}$$

Hope: $E'(t) \leq 0$

$$\delta\lambda^{2\delta}\underbrace{\dot{u}^2}_{x^2} + \delta\lambda^{2+2\delta}c(t)\underbrace{u^2}_{y^2} + \lambda^2 c(t)\underbrace{u\dot{u}}_{xy} \geq 0$$

General fact in lin. algebra:

$$Ax^2 + By^2 + Cxy \geq 0$$

for every $(x, y) \in \mathbb{R}^2$



$$4AB \geq C^2, \quad A \geq 0, B \geq 0$$

$$E'(t) \leq 0 \Leftrightarrow 4\delta^2\lambda^{2+4\delta}\cancel{c(t)} \geq \lambda^4 c(t)^2$$

$$\Leftrightarrow 4\delta^2 \geq \lambda^{2-4\delta} c(t)$$

ok if $\delta > \frac{1}{2}$ or $\delta = \frac{1}{2}$ $4\delta^2 \geq \mu_2$

In these cases we obtain $E_k(t) \leq E_k(0)$

$$\Rightarrow \lambda_k^{4\delta} u_k^2(t) \leq M (|u_{1k}|^2 + \lambda_k^{4\delta} u_{0k}^2)$$

Good estimate for $u_k(t)$.

$$\lambda_k^{4d} u_k^2(t) \leq M \left(\lambda_k^{4d} \frac{u_{1k}^2}{\lambda_k^{4\sigma}} + \lambda_k^{4d} u_{0k}^2 \right)$$

$$= M \left(\underbrace{\frac{\lambda_k^{4\beta} u_{1k}^2}{\lambda_k^{4\sigma+4\beta-4d}}}_{\text{exponent is } \geq 0} + \lambda_k^{4d} u_{0k}^2 \right)$$

exponent is ≥ 0

$$\Leftrightarrow 4\sigma + 4\beta - 4d \geq 0$$

$$\Leftrightarrow d - \beta \leq \sigma$$

If $d - \beta \leq \sigma$ we obtain

$$\sum_{k \geq 0} \lambda_k^{4d} u_k^2(t) \leq M \underbrace{\sum_{k \geq 0} \left(\lambda_k^{4\beta} u_{1k}^2 + \lambda_k^{4d} u_{0k}^2 \right)}_{\text{convergent because } (u_0, u_1) \in D(A^d) \times D(A^\beta)}$$

$$\Rightarrow u(t) \in D(A^d).$$

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Estimate for $\dot{u}_k$

$$\ddot{u}_k + 2\delta \lambda_k^{2\sigma} \dot{u}_k = -\lambda_k^2 c(t) u_k$$

See this as a first order linear equ. in $\dot{u}_k \rightsquigarrow$

$$\dot{u}_k(t) = u_{1k} e^{-2\delta \lambda_k^{2\sigma} t} - e^{-2\delta \lambda_k^{2\sigma} t} \int_0^t \underbrace{\lambda_k^2 c(s)}_{\leq \mu_2} u_k(s) e^{2\delta \lambda_k^{2\sigma} s} ds$$

$$|\dot{u}_k(t)| \leq |u_{1k}| + \lambda_k^2 \mu_2 \max_{[0,t]} |u_k(s)| e^{-2\delta \lambda_k^{2\sigma} t} \int_0^t e^{2\delta \lambda_k^{2\sigma} s} ds$$

$$\leq |u_{1k}| + \lambda_k^2 \mu_2 \max |u_k(s)| \frac{1}{2\delta \lambda_k^{2\sigma}}$$

$$u_k^2(t) \leq M \left(|u_{1k}|^2 + \lambda_k^4 \max |u_k(s)|^2 \frac{1}{\lambda_k^{4\sigma}} \right)$$

↓
use previous estimate

$$\leq M \left(|u_{1k}|^2 + \frac{\lambda_k^4}{\lambda_k^{4\sigma}} \left(\frac{u_{1k}^2}{\lambda_k^{4\sigma}} + u_{0k}^2 \right) \right)$$

$$\leq M \left(|u_{1k}|^2 + \frac{\lambda_k^4}{\lambda_k^{4\sigma}} u_{0k}^2 \right)$$

$$\lambda_k^{4\beta} u_k^2(t) \leq M \left(\lambda_k^{4\beta} u_{1k}^2 + \underbrace{\frac{\lambda_k^{4\beta} \lambda_k^4}{\lambda_k^{4\sigma}} \frac{\lambda_k^{4\beta}}{\lambda_k^{4\sigma}}}_{\frac{1}{\lambda_k^{4\sigma+4\beta-4\sigma-4}}} u_{0k}^2 \right)$$

$$\begin{aligned} \text{exponent} \geq 0 &\Leftrightarrow 2 - \beta - 1 + \sigma \geq 0 \\ &\Leftrightarrow 2 - \beta \geq 1 - \sigma \end{aligned}$$

As before we conclude that $u(t) \in D(A^\beta)$.

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Remark 1 We obtain also boundedness of high frequencies of u i.e.

$$D(A^\alpha) \times D(A^\beta)$$

This is because $E_k'(t) \leq 0$ for k large

Remark If $\sigma \in [\frac{1}{2}, 1)$, then $u(t)$ has Gevrey regularity for $t \geq 0$

How much regularity? $\varphi(\lambda) = \lambda^{2(1-\sigma)}$

Main idea for Gevrey regularity: improve

$$E_k'(t) \leq 0$$

to

$$E_k'(t) \leq - R \underbrace{c(t)}_{\substack{\uparrow \\ \text{needed}}} \lambda_k^{2(1-\sigma)} E_k(t)$$

$$\leadsto u(t) \in \underbrace{G_{\varphi, R G(t), 1/2}}_{\text{needed}} \quad \text{with} \quad G(t) := \int_0^t c(s) ds$$