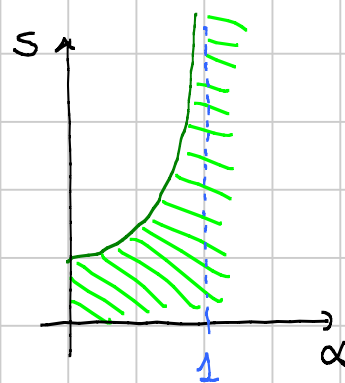


$$\ddot{u} + c(t) Au = 0$$

Well posedness. Assume

- $0 < \mu_1 \leq c(t) \leq \mu_2$
- $c(t)$ is ω -continuous
- $(u_0, u_1) \in \mathcal{G}_{\varphi, r, 1/2} \times \mathcal{G}_{\varphi, r, 0}$

$\frac{\lambda}{\varphi(\lambda)} \omega\left(\frac{1}{\lambda}\right)$ is bounded



Counterexample

Assume $c(t)$ as above

Assume (u_0, u_1) as above BUT $\lim_{\lambda \rightarrow \infty} \frac{\lambda}{\varphi(\lambda)} \omega\left(\frac{1}{\lambda}\right) = +\infty$

Then it may happen that $(u(t), \dot{u}(t))$ is extremely irregular for $t > 0$, namely

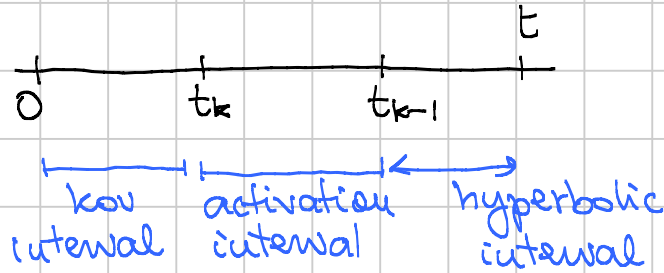
$$(u(t), \dot{u}(t)) \notin \mathcal{G}_{-\varphi, r, 1/2} \times \mathcal{G}_{-\varphi, r, 0}$$

with $\lim_{\lambda \rightarrow \infty} \frac{\lambda}{\varphi(\lambda)} \omega\left(\frac{1}{\lambda}\right) = +\infty$

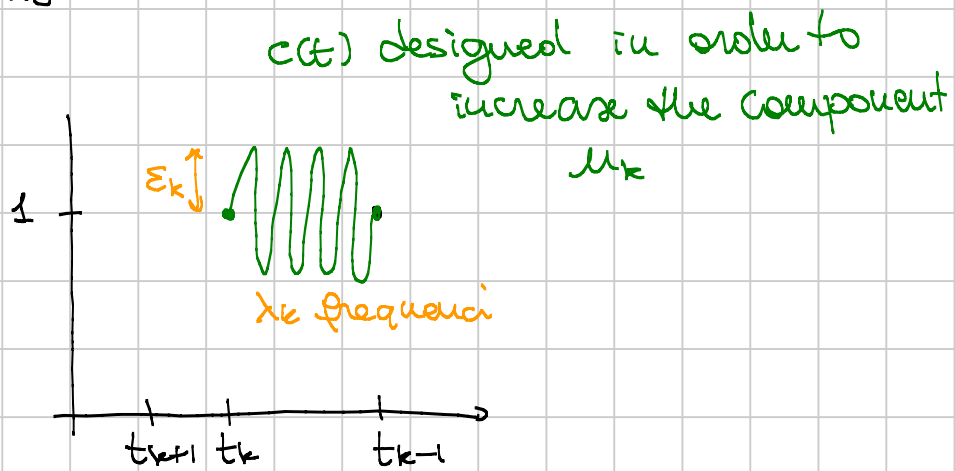
$$\ddot{u}_k + c(t) \lambda_k^2 u_k = 0$$

Want to find $c(t)$ in such a way that all components have growing energy.

Strategy



$c(t)$ is defined like this

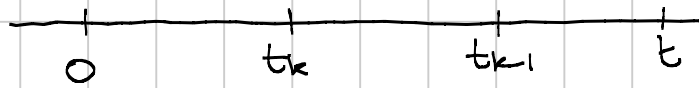


In each interval

- λ_k is the frequency ($\lambda_k \rightarrow +\infty$)
- ϵ_k is the amplitude of oscillations ($\epsilon_k \rightarrow 0$)

Hope: $c(t)$ globally is ω -continuous

Define $u_k(t)$ as the solution which grows a lot in the interval $[t_k, t_{k-1}]$



$E_k(t_k) \leadsto$ to be chosen

$$E_k(t_{k-1}) \sim E_k(t_k) \cdot \exp \{ \varepsilon_k \lambda_k (t_{k-1} - t_k) \}$$

↑
lecture 7

$$\sim E_k(t_k) \cdot \exp \{ \varepsilon_k \lambda_k t_{k-1} \}$$

↑
reasonable if $t_k \ll t_{k-1}$

$$E_k(0) \leq E_k(t_k) \exp \{ \lambda_k t_k \}$$

↑
kovalestian
estimate

$$E_k(t) \geq E_k(t_{k-1}) \exp \{ -\varepsilon_{k-1} \lambda_{k-1} t \}$$

↑
hyperbolic
estimate

Hyperbolic estimate: if $c(t)$ is of class C^1 , then the variation of the energy in a interval is less than

$$\exp \left\{ \int_{\text{interval}} \frac{|\dot{c}(s)|}{c(s)} ds \right\}$$

Final requirements

→ very regular at time $t=0$

$$\sum_{k \geq 0} E_k(0) \exp \{ 2k \varphi(\lambda_k) \} < +\infty \quad \forall \lambda > 0$$

$$\sum_{k \geq 0} E_k(0) \exp \{ 2k \varphi(\lambda_k) \} < +\infty$$

→ very irregular at time $t > 0$

$$\sum_{k \geq 0} E_k(t) \exp \{-2R \psi(\lambda_k)\} = +\infty \quad \forall R > 0$$

$$\sum_{k \geq 0} E_k(t) \exp \{-2k \psi(\lambda_k)\} = +\infty$$

→ $c(t)$ to be ω -continuous... equivalent to

$$\frac{\varepsilon_k}{\omega(\frac{1}{\lambda_k})} \text{ bounded}$$

Conditions are

$$\sum E_k(t_k) \exp \{ \lambda_k t_k + 2k \varphi(\lambda_k) \} < +\infty$$

$$\sum E_k(t_k) \exp \{ -\varepsilon_{k-1} \lambda_{k-1} t + \varepsilon_k \lambda_k t_{k-1} - 2k \psi(\lambda_k) \} = +\infty$$

$$\frac{\varepsilon_k}{\omega(\frac{1}{\lambda_k})} \text{ bounded}$$

$$\text{Set } t_k = \frac{2\pi}{\lambda_k} \quad \varepsilon_k = \omega\left(\frac{1}{\lambda_k}\right)$$

The first series becomes

$$\sum \underbrace{E_k(t_k)}_{\frac{1}{k^2}} \exp \{ 2\pi + 2k \varphi(\lambda_k) \} = \sum \exp \{-2 \log k\}$$

$$E_k(t_k) = \exp \{ -2 \log k - 2k \varphi(\lambda_k) - 2\pi \}$$

Guarantees the convergence of the first series

Need the divergence of

$$\sum E_k(t_k) \exp \left\{ -\varepsilon_{k-1} \lambda_{k-1} t + \varepsilon_k \lambda_k t_{k-1} - 2k \varphi(\lambda_k) \right\}$$
$$= \sum \exp \left\{ -2 \log k - 2k \varphi(\lambda_k) - 2\pi - \varepsilon_{k-1} \lambda_{k-1} t - 2k \varphi(\lambda_k) + \right.$$

$\left. + \varepsilon_k \lambda_k t_{k-1} \right\} = +\infty$

plus term has to win !!!

$$\varepsilon_k \lambda_k t_{k-1} = \omega\left(\frac{1}{\lambda_k}\right) \lambda_k \frac{2\pi}{\lambda_{k-1}}$$

$$\omega\left(\frac{1}{\lambda_k}\right) \lambda_k \frac{2\pi}{\lambda_{k-1}} \gg \log k$$

$$\omega\left(\frac{1}{\lambda_k}\right) \lambda_k \frac{2\pi}{\lambda_{k-1}} \gg k \varphi(\lambda_k)$$

$$\omega\left(\frac{1}{\lambda_k}\right) \lambda_k \frac{2\pi}{\lambda_{k-1}} \gg k \varphi(\lambda_k)$$

$$\omega\left(\frac{1}{\lambda_k}\right) \lambda_k \frac{2\pi}{\lambda_{k-1}} \gg t \lambda_{k-1} \omega\left(\frac{1}{\lambda_{k-1}}\right)$$

$$\frac{\lambda_k}{\varphi(\lambda_k)} \omega\left(\frac{1}{\lambda_k}\right) \gg k \lambda_{k-1}$$

Unbounded $\Rightarrow$
given λ_{k-1} I can
choose λ_k