

CounterexamplesBasic estimate

ODE

$$\ddot{u} + \lambda^2 c(t) u = 0$$

$$E(t) \leq E(0) \exp\{M \lambda t\}$$

$$\uparrow$$

$$\dot{u}^2 + \lambda^2 u^2$$

Basic tool

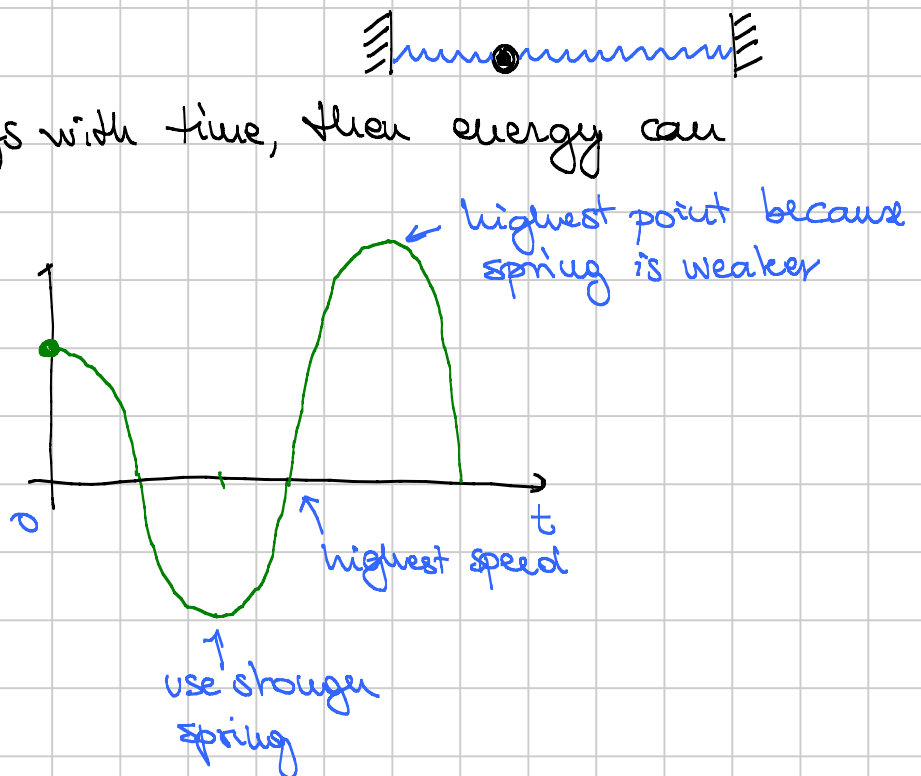
Showing that previous estimate is optimal: finding $c(t)$ and $u(t)$ for which we have equality.

Heuristics

$$\ddot{u} + \lambda^2 u = 0$$

$$u(t) = c_1 \cos(\lambda t) + c_2 \sin(\lambda t)$$

If we change the springs with time, then energy can grow exponentially.



Basic idea:

- use weaker spring when moving in the opposite dir. with respect to the origin
- use stronger spring toward the origin.

Idea Look for a solution of type

$$u(t) = \sin(\lambda t) e^{a(t)} \quad \uparrow \text{final possibly increasing with time}$$

$$\dot{u} = \lambda \cos(\lambda t) e^{a(t)} + \sin(\lambda t) \dot{a}(t) e^{a(t)}$$

$$\ddot{u} = -\lambda^2 \sin(\lambda t) e^{a(t)} + 2\lambda \cos(\lambda t) \dot{a}(t) e^{a(t)} + \sin(\lambda t) \ddot{a}(t) e^{a(t)} + \sin(\lambda t) \dot{a}(t)^2 e^{a(t)}$$

$$\ddot{u} + c(t) \lambda^2 u(t) = 0 \quad (\Rightarrow)$$

$$\begin{aligned} & -\lambda^2 \sin(\lambda t) + 2\lambda \cos(\lambda t) \dot{a}(t) + \sin(\lambda t) \ddot{a}(t) + \sin(\lambda t) \dot{a}(t)^2 + \\ & + c(t) \lambda^2 \sin(\lambda t) = 0 \end{aligned}$$

$$\sin(\lambda t) \{ -\lambda^2 + \ddot{a}(t) + \dot{a}(t)^2 + c(t) \lambda^2 \} + 2\lambda \cos(\lambda t) \dot{a}(t) = 0$$

Reasonable choice: choose $\dot{a}(t) = \sin(\lambda t)$. something
 $= \sin(\lambda t) \cdot \sin(\lambda t) \propto$
 $= \alpha \sin^2(\lambda t)$ ↑ constant

$$\ddot{a}(t) = 2\alpha \sin(\lambda t) \cos(\lambda t) \cdot \lambda$$

$$\leadsto -\lambda^2 + 2\alpha \lambda \sin(\lambda t) \cos(\lambda t) + \alpha^2 \sin^4(\lambda t) + c(t) \lambda^2 + \cancel{2\alpha \lambda \cos(\lambda t) \sin(\lambda t)} = 0$$

Choose $\alpha = \varepsilon \lambda$

$$\leadsto -\cancel{\lambda^2} + 4\varepsilon \cancel{\lambda^2} \sin(\lambda t) \cos(\lambda t) + \varepsilon^2 \cancel{\lambda^2} \sin^4(\lambda t) + c(t) \cancel{\lambda^2} = 0$$

$$\leadsto c(t) = 1 - 4\varepsilon \sin(\lambda t) \cos(\lambda t) - \varepsilon^2 \sin^4(\lambda t)$$

If ε is small enough, then $\frac{1}{2} \leq c(t) \leq \frac{3}{2}$

$$\leadsto \ddot{a}(t) = \varepsilon \lambda \sin^2(\lambda t)$$

$$\leadsto a(t) = \frac{1}{2} \varepsilon \lambda t - \frac{1}{4} \varepsilon \sin(2\lambda t)$$

$$\leadsto u(t) = \sin(\lambda t) \exp \left\{ \underbrace{\frac{1}{2} \varepsilon \lambda t - \frac{1}{4} \varepsilon \sin(2\lambda t)}_{\substack{\text{grows as } \lambda t \\ \text{when } t \rightarrow \infty}} \right\}$$

Remark $\rightarrow u(t)$ oscillates with "period" $\frac{2\pi}{\lambda}$
 $\rightarrow c(t)$ oscillates with period $\frac{E}{\lambda}$ $\left. \vphantom{\frac{E}{\lambda}} \right\} \frac{1}{2}$

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Consider now a damped equ

$$\ddot{u} + 2\delta \lambda^2 \dot{u} + \lambda^2 c(t) u = 0$$

As before

$$E(t) \leq E(0) \exp \{ M \lambda t \}$$

Question: is this optimal?

Let us look for a solution of the form

$$u(t) = \sin(\lambda t) e^{a(t)}$$

Compute $\dot{u}$ and $\ddot{u}$ as before:

$$\ddot{u} = \lambda \cos(\lambda t) e^{a(t)} + \sin(\lambda t) \dot{a}(t) e^{a(t)}$$

$$\ddot{u} = -\lambda^2 \sin(\lambda t) e^{a(t)} + 2\lambda \cos(\lambda t) \dot{a}(t) e^{a(t)} + \sin(\lambda t) \ddot{a}(t) e^{a(t)} + \sin(\lambda t) \dot{a}(t)^2 e^{a(t)}$$

$$\ddot{u} + 2\delta \lambda^{2\sigma} \dot{u} + \lambda^2 c(t) u = 0 \Leftrightarrow$$

$$-\lambda^2 \sin(\lambda t) + 2\lambda \cos(\lambda t) \dot{a} + \sin(\lambda t) \ddot{a} + \sin(\lambda t) \dot{a}^2 +$$

$$2\delta \lambda^{2\sigma+1} \cos(\lambda t) + 2\delta \lambda^{2\sigma} \sin(\lambda t) \dot{a} + \lambda^2 c(t) \sin(\lambda t) = 0$$

$$\sin(\lambda t) \{ -\lambda^2 + \ddot{a} + \dot{a}^2 + 2\delta \lambda^{2\sigma} \dot{a} + \lambda^2 c(t) \} +$$

$$+ \cos(\lambda t) \{ 2\lambda \dot{a} + 2\delta \lambda^{2\sigma+1} \} = 0$$

Reasonable choice :

$$2\lambda \dot{a}(t) = -2\delta \lambda^{2\sigma+1} + \sin(\lambda t) \cdot \varepsilon \lambda \sin(\lambda t)$$

$$\text{namely } \ddot{a} = -\delta \lambda^{2\sigma} + \frac{1}{2} \varepsilon \lambda \sin^2(\lambda t)$$

$$\leadsto \ddot{a} = \varepsilon \lambda^2 \sin(\lambda t) \cos(\lambda t)$$

$$\leadsto \sin(\lambda t) \{ -\lambda^2 + \varepsilon \lambda^2 \sin(\lambda t) \cos(\lambda t) + \delta^2 \lambda^{4\sigma-2} + \frac{1}{4} \varepsilon^2 \lambda^2 \sin^4(\lambda t) +$$

$$- \delta \lambda^{2\sigma+1-1} \varepsilon \sin^2(\lambda t) - 2\delta^2 \lambda^{4\sigma-2} + \delta \varepsilon \lambda \sin^2(\lambda t) + \lambda^2 c(t) \}$$

$$+ \cos(\lambda t) \varepsilon \lambda^2 \sin(\lambda t) \cos(\lambda t) = 0$$

$$\leadsto c(t) = 1 - 2\varepsilon \sin(\lambda t) \cos(\lambda t) + \varepsilon \delta \lambda^{2\sigma-1} \sin^2(\lambda t) \\ + \delta \lambda^{2\sigma-1} \varepsilon \sin^2(\lambda t) + \delta^2 \lambda^{4\sigma-2}$$

If ε is small enough, then $\frac{1}{2} \leq c(t) \leq \frac{3}{2}$

$$4\sigma - 2 \geq 0 \quad (\Rightarrow) \quad \sigma \geq \frac{1}{2}$$

$$a(t) = \underbrace{(2\varepsilon \lambda - \delta \lambda^{2\sigma})}_{\uparrow} t - \varepsilon \sin(2\lambda t)$$

if $2\sigma < 1$ ($\sigma < \frac{1}{2}$)
or $2\sigma = 1$ and $\delta \ll 1$
then $a(t)$ grows as λt