

Equations with time-dependent coefficients: classical theory

(Rephrasing of De Giorgi - Colombini - Spagnolo [Ann. SNS 1979] in modern language)

$$\ddot{u} + \underbrace{\delta(t)}_{\geq 0} \dot{u} + c(t) Au = 0$$

$\delta \equiv 0$ is ok

$$A : D(A) \rightarrow H \quad \text{with } D(A) \subseteq H$$

As in Lecture 1 $\exists \{e_k\} \subseteq H$ orth. syst. $\exists \{\lambda_k\} \subseteq [0, +\infty)$ s.t.

$$Ae_k = \lambda_k^2 e_k$$

Fact Any $u : [0, T] \rightarrow H$ can be identified with the sequence of components

$$u_k(t) := \langle u(t), e_k \rangle$$

Original equ. is equivalent to system of ODEs

$$\ddot{u}_k + \delta(t) \dot{u}_k + \lambda_k^2 c(t) u_k = 0$$

"Space regularity" of $u \rightsquigarrow u(t) \in D(A^\alpha)$

$$\in D(A^{-\alpha})$$

$$\in \mathcal{G}_{\varphi, r, \alpha}(A)$$

$$\in \mathcal{G}_{-\psi, R, \alpha}(A)$$

Sufficient condition for $u \in C^0([0, T], D(A^{\alpha}))$ is that

$$\sum_{k \geq 0} (1 + \lambda_k)^{4\alpha} \sup \{ |u_k(t)|^2 : t \in [0, T] \}$$

and $u_k(t)$ continuous for every $k \in \mathbb{N}$

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① Easy case Assume $c(t) \in C^1([0, T])$
 $c(t) \geq \mu_1 > 0$ (S.H.)

Then the problem

$$\ddot{u} + \underbrace{\delta(t)}_{\geq 0} \dot{u} + c(t) A u = 0$$

$$u(0) = u_0 \quad \dot{u}(0) = u_1$$

is well-posed in $D(A^{1/2}) \times H$ (or more generally $D(A^{\alpha+1/2}) \times D(A^{\alpha})$)

$$(u_0, u_1) \in D(A^{1/2}) \times H \Rightarrow$$

$$u \in C^0([0, T], D(A^{1/2})) \cap C^1([0, T], H)$$

Proof. Lecture 2 $\Rightarrow \exists M$ indep. of k s.t.

$$\dot{u}_k(t)^2 + \lambda_k^2 u_k(t)^2 \leq \underbrace{(\dot{u}_{1k}^2 + \lambda_k^2 u_{0k}^2)}_{\text{The series Bounded if}} \exp(\mu_1 t)$$

(u_0, u_1) $\in D(A^{1/2}) \times H$

$\Rightarrow$ supremum of LHS in $[0, T]$ gives a converging series.

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② WORST CASE $c(t) \in L^1([0, T])$
NO SIGN CONDITION

Then the problem is well-posed in

$$\mathcal{G}_{-\psi, R, 1/2}(A) \times \mathcal{G}_{-\psi, R, 0}(A)$$

provided that $\psi(\sigma) = \sigma$ (analytic ultradistribution)

More precisely, assume that $(u_0, u_1) \in \mathcal{G}_{-\psi, R_0, 1/2} \times \mathcal{G}_{-\psi, R_0, 0}$
then there exists

$$R: [0, T] \rightarrow (0, +\infty)$$

with

$$R(0) = R_0$$

such that

$$u(t) \in \mathcal{G}_{-\psi, R(t), 1/2} \times \mathcal{G}_{-\psi, R(t), 0}$$

Proof. We proved that

$$\begin{aligned} \dot{u}_k^2(t) + \lambda_k^2 u_k^2(t) &\leq (u_{1k}^2 + \lambda_k^2 u_{0k}^2) \exp\left(\lambda_k t + \lambda_k \int_0^t |c(s)| ds\right) \\ &= \underbrace{(\dots) \exp(-2\psi(\lambda_k) R_0)}_{\text{convergent series by assumption on } (u_0, u_1)} \underbrace{\exp(2\psi(\lambda_k) R_0 + \lambda_k t + \lambda_k C'(t))}_{\exp(\lambda_k \underbrace{(2R_0 + t + C'(t))}_{R(t)})} \end{aligned}$$

If we want a convergent series we need to multiply by

$$\exp(-\lambda_k R(t))$$

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Remark Cauchy-Kow theory.

$$\textcircled{3} \quad c(t) \geq \mu_1 > 0 \quad c(t) \in L^1([0, T])$$

The pbm. is well posed in $\mathcal{G}_{\varphi, \nu, 1/2} \times \mathcal{G}_{\varphi, \nu, 0}$
with $\varphi(\lambda) = \lambda$

Proof. Lecture 2 $\leadsto \exists M_1 \quad \forall \varepsilon > 0 \exists M_\varepsilon$ s.t.

$$\underbrace{u_k^2(t) + \lambda_k^2 u_k^2(t)}_{\text{LHS}_k} \leq (u_{ik}^2 + \lambda_k^2 u_{0k}^2) \exp\{M_1 \varepsilon \lambda_k t + M_\varepsilon\}$$

Choose $\varepsilon > 0$ small enough so that $M_1 \varepsilon T = r_1 < r_0$
 $\uparrow$
radius of
initial conditions

$$\sum \text{LHS}_k \exp((r_0 - r_1) \lambda_k) \leq \sum (u_{ik}^2 + \lambda_k^2 u_{0k}^2) \exp((r_0 - r_1) \lambda_k + M_1 \varepsilon \lambda_k t + M_\varepsilon)$$

$$\begin{aligned} & \text{convergent} \\ & = \sum (u_{ik}^2 + \lambda_k^2 u_{0k}^2) \exp(r_0 \lambda_k) \\ & \quad \exp(\underbrace{-r_1 \lambda_k + M_1 \varepsilon \lambda_k t + M_\varepsilon}_{\leq 0}) \end{aligned}$$

$$\textcircled{4} - \textcircled{5} \quad c(t) \geq \mu_1 > 0$$

$c(t)$ is ω -continuous in $[0, T]$

Assume that $\frac{\lambda_k}{\varphi(\lambda_k)} \omega\left(\frac{1}{\lambda_k}\right)$ is bounded (wrt k)

Then the pbm is well posed

both in $\mathcal{G}_{\varphi, \nu, 1/2} \times \mathcal{G}_{\varphi, \nu, 0}$

and in $\mathcal{G}_{-\varphi, R, 1/2} \times \mathcal{G}_{-\varphi, R, 0}$

Proof. Lecture 2 $\leadsto \exists M$ (indep of everything but for μ) such that

$$\underbrace{\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t)}_{LHS_k} \leq (\dot{u}_{0k}^2 + \lambda_k^2 u_{0k}^2) \exp\left\{\lambda_k \omega\left(\frac{1}{\lambda_k}\right) M t\right\}$$

Consider now initial conditions in $G_{\varphi, r_0, 1/2} \times G_{\varphi, r_0, 0}$

$$\sum LHS_k \exp\left\{+2\varphi(\lambda_k) \underbrace{(r_0 - R t)}_{\substack{\uparrow \\ \text{decay of the radius}}}\right\} \leq$$

$$\leq \underbrace{\sum (\dot{u}_{0k}^2 + \lambda_k^2 u_{0k}^2)}_{\text{convergent}} \exp\left\{2\varphi(\lambda_k) r_0 - 2\varphi(\lambda_k) R t + \lambda_k \omega\left(\frac{1}{\lambda_k}\right) M t\right\}$$

Hope: choosing R so that this is bounded from above

$$-2\varphi(\lambda_k) R + \lambda_k \omega\left(\frac{1}{\lambda_k}\right) M \leq 0$$

$$\underbrace{\frac{\lambda_k}{\varphi(\lambda_k)} \omega\left(\frac{1}{\lambda_k}\right) M}_{\substack{\text{by assumption} \\ \text{this is bounded}}} \leq 2R$$

Remark Solution exists as long as $r_0 - R t > 0$
 $(\Rightarrow) t < \frac{r_0}{R}$

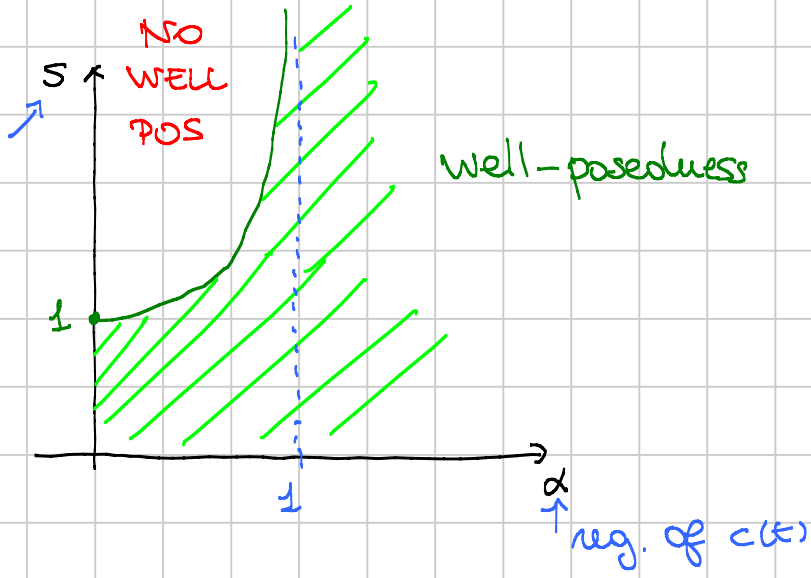
Proof of ⑤ is analogous.

Special case $\omega(\sigma) = M \sigma^\alpha \leadsto \frac{\lambda_k}{\varphi(\lambda_k)} \frac{1}{\lambda_k^\alpha}$ is bounded

$\varphi(\lambda_k) = \lambda_k^{1-\alpha} \leadsto$ Gevrey space with $S = \frac{1}{1-\alpha}$

Picture

space
reg.



Weak hyper

