

Approximated energy estimates

Consider a family of ODEs

$$\ddot{u}_\lambda + \boxed{\delta(t)} \dot{u}_\lambda + \lambda^2 c(t) u_\lambda = 0$$

$\delta(t) \geq 0$ parameter

Goal: estimating growth of $u(t)$ as $t \rightarrow +\infty$ depending on λ

ENERGY ESTIMATES

① Stupid energy $S(t) := \dot{u}^2(t) + u^2(t)$

$$S'(t) := 2\dot{u}\ddot{u} + 2u\dot{u}$$

$$= 2\dot{u}(-\delta(t)\dot{u} - \lambda^2 c(t)u) + 2u\dot{u}$$

$$= \underbrace{-2\delta(t)\dot{u}^2}_{\leq 0} + \underbrace{(-\lambda^2 c(t) + 1)}_{\leq 1 + \lambda^2 |c(t)|} \underbrace{2u\dot{u}}_{\leq \dot{u}^2 + u^2 = S(t)}$$

$$\leq (1 + \lambda^2 |c(t)|) S(t)$$

$$S(t) \leq S(0) \exp \left\{ \underbrace{\int_0^t (1 + \lambda^2 |c(s)|) ds}_{\sim \exp(\lambda^2 t)} \right\}$$

① KOVALESKIAN ENERGY

$$E(t) := \dot{u}^2 + \lambda^2 u^2$$

$$\begin{aligned} E'(t) &= 2\dot{u}\ddot{u} + 2\lambda^2 u\dot{u} \\ &= 2\dot{u}(-\delta(t)\dot{u} - \lambda^2 c(t)u) + 2\lambda^2 u\dot{u} \\ &= \underbrace{-2\delta(t)\dot{u}^2}_{\leq 0} + \lambda \underbrace{(1-c(t))}_{\leq 1+|c(t)|} \underbrace{2\lambda u\dot{u}}_{\leq \dot{u}^2 + \lambda^2 u^2} \\ &= E(t) \end{aligned}$$

$$\leq \lambda (1+|c(t)|) E(t)$$

↑
λ instead of λ²

② HYPERBOLIC ENERGY

$$F(t) := \dot{u}^2 + \lambda^2 \delta(t) u^2$$

↑
class C¹, nonnegative (or strictly pos.)

$$F'(t) = 2\dot{u}(-\delta(t)\dot{u} - \lambda^2 c(t)u) + 2\lambda^2 \delta(t)u\dot{u} + \lambda^2 \dot{\delta}(t)u^2$$

$$\begin{aligned} &= \underbrace{-2\delta(t)\dot{u}^2}_{\leq 0} + \lambda \frac{(\delta(t)-c(t))}{\sqrt{\delta(t)}} \underbrace{2\sqrt{\delta(t)}\lambda u\dot{u}}_{\leq \dot{u}^2 + \lambda^2 \delta(t)u^2} + \underbrace{\frac{\dot{\delta}(t)}{\delta(t)} \lambda^2 u^2}_{\leq F(t)} \\ &\leq \frac{|\delta(t)-c(t)|}{\sqrt{\delta(t)}} \lambda + \frac{|\dot{\delta}(t)|}{\delta(t)} \lambda^2 u^2 \\ &= F(t) \end{aligned}$$

$$F'(t) \leq \left(\frac{|\delta(t)-c(t)|}{\sqrt{\delta(t)}} \lambda + \frac{|\dot{\delta}(t)|}{\delta(t)} \lambda^2 \right) F(t)$$

Standard applications

① $c(t) \in L^1((0, T))$ for every $T > 0$. No sign condition

Kowalewskian energy $\leadsto$

$$E(t) \leq E(0) \exp \left\{ \lambda \underbrace{\int_0^t (1 + |c(s)|) ds}_{\text{convergent}} \right\}$$

$$= E(0) \exp \left\{ \lambda t + \lambda \int_0^t |c(s)| ds \right\}$$

② $c(t)$ Lipschitz cont. and $c(t) \geq \mu_1 > 0$ (STRICT HYPERB.)

Assume for a while that $c \in C^1$. Take $\delta(t) = c(t)$ in the hyp. energy and obtain

$$F'(t) \leq \frac{|c'(t)|}{c(t)} F(t) \leq \frac{L}{\mu_1} F(t) \stackrel{\text{Lip. const.}}{=} M \cdot F(t)$$

$$F(t) \leq F(0) \exp \{ M t \}$$

$\uparrow$ No λ here!!!

③ $c \in L^1((0, T))$ $c(t) \geq \mu_1 > 0$ (S.H.)

Take $\delta(t)$ such that $\|\delta(t) - c(t)\|_{L^1} \leq \varepsilon$, $\delta \in C^1$ and use hyp. energy

$\downarrow$ (assume that $\delta(t) \geq \mu_1 > 0$)

$$F(t) \leq F(0) \exp \left\{ \lambda \int_0^t \frac{|\delta(s) - c(s)|}{\sqrt{\delta(s)}} ds + \int_0^t \frac{|\delta'(s)|}{\delta(s)} ds \right\}$$

$$= F(0) \exp \left\{ \frac{\lambda \varepsilon}{\mu_1} + M_\varepsilon t \right\}$$

$\uparrow$ indep. of λ

④ $c(t) \geq \mu_1 > 0$ (s.h.) + $c(t)$ ω -continuous.

For every $\varepsilon > 0$ take $\delta(t) := c_\varepsilon(t) = \frac{1}{\varepsilon} \int_t^{t+\varepsilon} c(s) ds$

Easy estimates:

$$(1) |c_\varepsilon'(t)| \leq \frac{\omega(\varepsilon)}{\varepsilon}$$

$$(2) |c(t) - c_\varepsilon(t)| \leq \omega(\varepsilon)$$

Use the hyperbolic energy with $\delta(t) := c_\varepsilon(t)$

$$F'(t) \leq \left(\frac{|c(t) - \delta(t)|}{\sqrt{\delta(t)}} \lambda + \frac{|\delta'(t)|}{\delta(t)} \right) F(t)$$

$$\leq \left(\frac{\omega(\varepsilon)}{\sqrt{\mu_1}} \lambda + \frac{\omega(\varepsilon)}{\varepsilon} \frac{1}{\mu_1} \right) F(t)$$

KEY IDEA : choose $\varepsilon := \frac{1}{\lambda}$

[De Giorgi, Colombini, Spagnolo
SNS 1979]

$$F'(t) \leq k \left(\lambda \omega\left(\frac{1}{\lambda}\right) + \lambda \omega\left(\frac{1}{\lambda}\right) \right) F(t)$$

$$= 2k \lambda \omega\left(\frac{1}{\lambda}\right) F(t)$$

$$F(t) \leq F(0) \exp \left\{ 2k \lambda \omega\left(\frac{1}{\lambda}\right) t \right\}$$

Example Assume $c(t)$ is α -Hölder ... $\omega(x) = H x^\alpha$

$$F(t) \leq F(0) \exp \left\{ \text{const } \lambda^{1-\alpha} t \right\}$$

↑
better than λ

Proof of (1) and (2)

$$(1) \quad c'_\varepsilon(t) = \frac{1}{\varepsilon} (c(t+\varepsilon) - c(t))$$

$$|c'_\varepsilon(t)| = \frac{1}{\varepsilon} |c(t+\varepsilon) - c(t)| \leq \frac{\omega(\varepsilon)}{\varepsilon}$$

$$(2) \quad |c_\varepsilon(t) - c(t)| = \left| \frac{1}{\varepsilon} \int_t^{t+\varepsilon} c(s) ds - c(t) \right|$$

$$= \left| \frac{1}{\varepsilon} \int_t^{t+\varepsilon} (c(s) - c(t)) ds \right|$$

$$\leq \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \underbrace{|c(s) - c(t)|}_{\omega(\varepsilon)} ds$$

$$= \omega(\varepsilon).$$