

FUNCTIONAL SPACES, CONTINUITY MODUL

Fix notation

H is a real Hilbert space ($H := L^2(\Omega)$, with $\Omega \subseteq \mathbb{R}^n$)

H is separable

Def. A orthonormal system is a subset $\{e_k\}_{k \in \mathbb{N}} \subseteq H$ such that

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

Def. Given any $u \in H$, set $u_k := \langle u, e_k \rangle$ and

$$\sum_{k \in \mathbb{N}} u_k^2 = \|u\|^2 < +\infty$$

Advantage: we can identify vectors $u \in H$ with sequence $\{u_k\}$

$$\sum_{k \geq 0} u_k^2 < +\infty$$

Multiplication operators

$A: D(A) \rightarrow H$ if there exist a sequence $\{\lambda_k\} \subseteq \mathbb{R}$ (most of the time $\lambda_k \geq 0$)
 $\{e_k\}$ orth. system

$$u = \sum_{k \geq 0} u_k e_k \quad \rightsquigarrow \quad Au = \sum_{k \geq 0} \lambda_k^2 u_k e_k$$

well defined when

$$u \in D(A) := \left\{ u \in H : \sum_{k \geq 0} \lambda_k^4 u_k^2 < +\infty \right\}$$

Examples

$$H := L^2((0, \pi))$$

$$Au = -u_{xx} \quad \text{with}$$

DBC

NBC

PBC

In this A is a mult. operator with

$$\lambda_k = k$$

$k \geq 1$ if DBC

$k \geq 0$ if NBC or PBC

(coercive operator)

Similar situation if $\Omega \subset \mathbb{R}^n$, $A = -\Delta$, Ω open, bounded
and $\partial\Omega$ smooth enough.

Multiplication operator in general

A be any self-adjoint nonnegative operator in H

$$\langle Au, v \rangle = \langle u, Av \rangle \quad \Rightarrow \quad \langle Au, u \rangle \geq 0$$

Result We can identify H with $L^2(M, \mu)$

measure
space

measure on M

$$H \ni u$$

$\rightsquigarrow$

$$u(\xi) \in L^2(M, \mu)$$

$\downarrow$

$\downarrow$

$$Au$$

$$\lambda^2(\xi) u(\xi) \in L^2(M, \mu)$$

$$D(A) \rightsquigarrow \left\{ u(\xi) \in L^2(M, \mu) : \int_M \lambda^4(\xi) u^2(\xi) d\mu < +\infty \right\}$$

Standard example

$$H := L^2(\mathbb{R}^n)$$

$$A = -\Delta$$

$u(\xi)$ is the Fourier transform

Functional spaces

$\{e_k\}$ orthon system

$\{\lambda_k^2\}$ seq. of eigen. of A

$$u = \sum_{k \geq 0} u_k e_k$$

↑
comp. of u

$$Au = \sum_{k \geq 0} \lambda_k^2 u_k e_k$$

Def. (Spaces with respect to the operator A)

① Sobolev spaces Given any $\alpha \geq 0$ ($\alpha \in \mathbb{R}$)

$$D(A^\alpha) := \left\{ \sum_{k \geq 0} u_k e_k : \sum_{k \geq 0} (1 + \lambda_k)^{4\alpha} u_k^2 < +\infty \right\}$$

$$A^\alpha u := \sum_{k \geq 0} \lambda_k^{2\alpha} u_k e_k$$

② Distributions Given any $\alpha \geq 0$ ($\alpha \in \mathbb{R}$)

$$D(A^{-\alpha}) := \left\{ \sum_{k \geq 0} u_k e_k : \sum_{k \geq 0} (1 + \lambda_k)^{-4\alpha} u_k^2 < +\infty \right\}$$

③ Gevrey spaces Given any $\alpha \geq 0$

Given $r > 0$ (radius)

Given $\varphi: [0, +\infty) \rightarrow (0, +\infty)$

$$G_{\varphi, r, \alpha}(A) := \left\{ \sum_{k \geq 0} u_k e_k : \sum_{k \geq 0} (1 + \lambda_k)^{4\alpha} \underbrace{e^{2r\varphi(\lambda_k)}}_{\text{huge weight}} u_k^2 < +\infty \right\}$$

Examples

• $\varphi(\lambda) = \lambda$, $\alpha = 0$ $\leadsto$ analytic functions with power series converging with radius $= 1$.

• $\varphi(\lambda) = \lambda^{\frac{1}{s}}$, $\alpha = 0$ $\leadsto$ Standard Gevrey space,
($s > 1$) $s \in (0, 1)$

④ Gevrey ultra distributions

$$\alpha \geq 0$$

$$R > 0$$

$$\psi : [0, +\infty) \rightarrow (0, +\infty)$$

$$\mathcal{G}_{-\psi, R, \alpha}(A) := \left\{ \sum_{k \geq 0} \mu_k e_k : \sum_{k \geq 0} (1 + \lambda_k)^{4\alpha} e^{-2R\psi(\lambda_k)} \mu_k^2 < +\infty \right\}$$

Basic inclusions For every admissible value of the parameters
(assume also that $\varphi(\lambda) \rightarrow +\infty$ as $\lambda \rightarrow +\infty$
 $\psi(\lambda) \rightarrow +\infty$ as $\lambda \rightarrow +\infty$)

It turns out that

$$\mathcal{G}_{\varphi, r, \alpha}(A) \subseteq D(A^+) \subseteq H \subseteq D(A^{-\alpha}) \subseteq \mathcal{G}_{-\psi, r, -\alpha}(A)$$

All inclusions are strict if $\{\lambda_k\}$ is unbounded

$$\mathcal{G}_{\varphi, r_2, \alpha}(A) \subseteq \mathcal{G}_{\varphi, r_1, \alpha}(A) \quad r_2 > r_1$$

$$\mathcal{G}_{-\psi, R_2, \alpha}(A) \supseteq \mathcal{G}_{-\psi, R_1, \alpha}(A) \quad R_2 > R_1$$

General philosophy

$$\sum_{k \geq 0} \underbrace{w_k}_{\substack{\uparrow \\ \text{weight dep. on } k}} \mu_k^2 < +\infty$$

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Def. (Continuity modulus) Let (X, d_X) and (Y, d_Y) be metric spaces (for example $\mathbb{R}$ with standard metric).

Let $\omega : [0, +\infty) \rightarrow [0, +\infty)$ be a function

Let $c : X \rightarrow Y$ be any function.

We say that c is ω -continuous in X if

$$d_Y(f(b), f(a)) \leq \omega(d_X(b, a)) \quad \forall (a, b) \in X^2$$

Examples

- $f: \mathbb{R} \rightarrow \mathbb{R} \quad \omega(\sigma) = L|\sigma|$

$$|f(b) - f(a)| \leq L |b - a| \quad \forall (a, b) \in \mathbb{R}^2$$

$\uparrow$
Lipschitz continuity

- $\omega(\sigma) = H|\sigma|^\alpha \quad \alpha \in (0, 1)$

$$|f(b) - f(a)| \leq H |b - a|^\alpha \quad \forall (a, b) \in \mathbb{R}^2$$

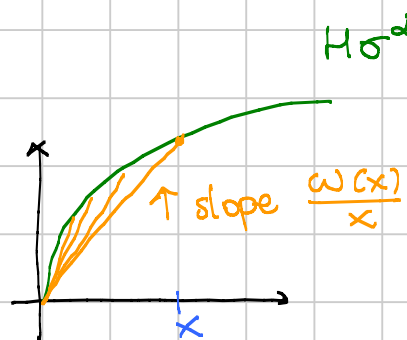
$\uparrow$
 α -Hölder continuity

Always assume that

(w1) $\omega(0) = 0$

(w2) ω is $\nearrow$

(w2) $\frac{\omega(x)}{x}$ is $\searrow$



(easy if ω is concave)

Remark Any cont. function on a compact space has a continuity modulus.

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