

Condition (F) $F(u) = \int_a^b L(x, u, \dot{u}) dx$

$$L: [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$$

$u_0: [a, b] \rightarrow \mathbb{R}$ solution to Euler equ.

We say that u_0 satisfies (F) if there exists $u: (-\varepsilon_0, \varepsilon_0) \times [a, b] \rightarrow \mathbb{R}$ smooth enough (C^2 is more than enough) such that

(i) $u(0, x) = u_0(x) \quad \forall x \in [a, b]$

(ii) $u_\varepsilon(\varepsilon, x) > 0 \quad \forall (\varepsilon, x) \in (-\varepsilon_0, \varepsilon_0) \times [a, b]$

(iii) $\forall \varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the function $x \rightarrow u(\varepsilon, x)$ is a solution to Euler equ.

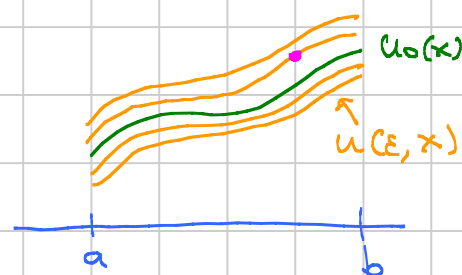
Let Ω be the image of u ($\Omega \subseteq \mathbb{R}^2$)

The slope function

$$p: \Omega \rightarrow \mathbb{R}$$

defined in such a way that

$$p(x, u(\varepsilon, x)) = u_\varepsilon(\varepsilon, x)$$



$$p(x, s)$$

$$(x, s) \in \Omega$$

(WRF) For every $v: [a, b] \rightarrow \mathbb{R}$ with $(x, v(x)) \in \Omega$ for every $x \in [a, b]$ and

$$v(a) = u_0(a)$$

$$v(b) = u_0(b)$$

It turns out that

$$F(v) - F(u_0) = \int_a^b E(x, v(x), p(x, v(x)), \dot{v}(x)) dx$$

where $E(x, s, p, q) := L(x, s, q) - L(x, s, p) - (q - p) L_p(x, s, p)$

MAIN RESULT

$$(F) \Rightarrow (WRF)$$

Two strategies : $\rightarrow$ à la Hilbert (via null Lagrangian)
 $\rightarrow$ à la Weierstrass.
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Proof à la Hilbert

First step: rewriting of Euler equ. in terms of slope function

$$p(x, u(\varepsilon, x)) = u_x(\varepsilon, x)$$

Let us derive wrt x :

$$p_x(x, u(\varepsilon, x)) + p_s(x, u(\varepsilon, x)) u_x(\varepsilon, x) = u_{xx}(\varepsilon, x)$$

$$p_x(x, u(\varepsilon, x)) + p_s(x, u(\varepsilon, x)) p(x, u(\varepsilon, x)) = u_{xx}(\varepsilon, x)$$

$$p_x + p_s p = u_{xx}$$

Euler equ: $\frac{d}{dx} L_p(x, u, \dot{u}) = L_s(x, u, \dot{u})$

$$L_{px}(x, u(\varepsilon, x), u_x(\varepsilon, x)) + L_{ps}(x, u(\varepsilon, x), u_x(\varepsilon, x)) u_x(\varepsilon, x) + \\ + L_{pp}(\dots) \underline{u_{xx}(\varepsilon, x)} = L_s(\dots)$$

$$L_{px}(x, u(\varepsilon, x), p(x, u(\varepsilon, x))) + L_{ps}(x, u(\varepsilon, x), p(x, u(\varepsilon, x))) p(x, u(\varepsilon, x)) + \\ + L_{pp}(\dots) (p_x + p_s p) = L_s(\dots)$$

$$L_{px}(x, s, p(x, s)) + L_{ps}(x, s, p(x, s)) p(x, s) + \\ L_{pp}(\dots) (p_x + p_s p) = L_s(\dots)$$

$$\boxed{L_{px} + L_{ps} p + L_{pp} p_x + L_{pp} p_s p = L_s} \quad \forall (x, s) \in \Omega.$$

(x, s)
 $(x, s, p(x, s))$

Second step: rewriting of (WRF).

$$F(v) - F(u_0) = \int_a^b L(x, v, \dot{v}) - \int_a^b L(x, u_0, \dot{u}_0)$$

$$\int_a^b E(x, v, p(x, v), \dot{v}) = \int_a^b L(x, v, \dot{v}) - \int_a^b L(x, v, p(x, v)) - \int_a^b (\dot{v} - p(x, v)) L_p(x, v, p(x, v))$$

$$(WRF) \Leftrightarrow \underbrace{\int_a^b L(x, u_0, \dot{u}_0) - L(x, v, p(x, v)) - (\dot{v} - p(x, v)) L_p(\dots)}_{\hat{F}(v)} = 0$$

$$\hat{F}(v) = \int_a^b \hat{L}(x, v, \dot{v}) dx \quad \text{where}$$

$$\hat{L}(x, s, p) := \underbrace{L(x, u_0(x), \dot{u}_0(x))}_{B(x, s)} - \underbrace{L(x, s, p(s, x))}_{A(x, s)} + p(x, s) L_p(x, s, p(s, x))$$

"Clearly" it turns out that $\hat{F}(u_0) = 0$ ($\dot{u}_0(x) = p(x, u_0(x))$)

Therefore (WRF) is proven if we show that $\hat{L}$ is a null Lagrangian, namely if

$$A_x = B_s$$

Third step: $\hat{L}$ is a null Lagrangian

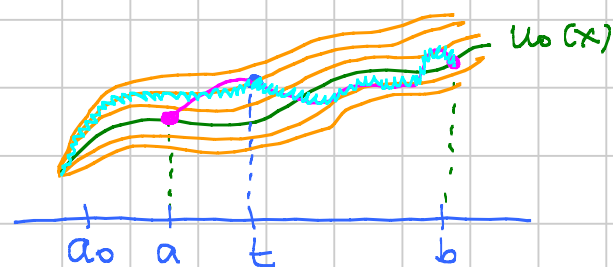
$$A_x = -L_{px} - L_{pp} p_x$$

$$B_s = -L_s - \cancel{L_{ps}} + \cancel{p_s L_p} + p L_{ps} + p L_{pp} p_s$$

$$A_x = B_s \Leftrightarrow \underbrace{\quad}_0 \underbrace{\quad}_0 \underbrace{\quad}_0 \text{ Euler equ. wrt slope function.}$$

PROOF À LA WEIERSTRASS

In the special case where the field originates from some point $a_0 < a$



Take v as before.

Take any point $t \in [a, b]$.

Consider $\varepsilon(t)$ in such a way that

$$u(\varepsilon(t), t) = v(t)$$

Consider $w(x)$ defined as
$$\begin{cases} u(\varepsilon(t), x) & x \in [a_0, t] \\ v(x) & x \in [t, b] \end{cases}$$

and define

$$f(t) := \int_{a_0}^b L(x, w, \dot{w}) dx$$

$$f(a) = \int_{a_0}^a L(x, u_0, \dot{u}_0) + \int_a^b L(x, v, \dot{v}) dx$$

$$f(b) = \int_{a_0}^a L(x, u_0, \dot{u}_0) + \int_a^b L(x, u_0, \dot{u}_0) dx$$

Therefore

$$F(v) - F(u_0) = f(a) - f(b) = - \int_a^b f'(x) dx$$

Claim: $-f'(x) = E(x, v(x), p(x, v(x)), \dot{v}(x))$

$$f(t) = \int_{a_0}^t L(x, u(\varepsilon(t), x), u_x(\varepsilon(t), x)) dx + \int_t^b L(x, v(x), \dot{v}(x)) dx$$

$f'(t)$ has many terms...

$$-L(t, v(t), \dot{v}(t)) \quad (\text{second integral})$$

$$L(t, u(\varepsilon(t), t), u_x(\varepsilon(t), t)) = L(t, v(t), p(t, v(t)))$$

$$\underbrace{\int_{a_0}^t L_S(\dots) u_\varepsilon \cdot \varepsilon_t}_{\text{integrate by parts}} + \int_{a_0}^t L_p(\dots) u_{x\varepsilon} \varepsilon_t dx$$

$$= - \underbrace{\int_{a_0}^t \varepsilon_t \frac{d}{dx} L_p(\dots) u_\varepsilon}_{=0 \text{ by Euler eqn.}}$$

$$+ \left[L_p(\dots) u_\varepsilon \varepsilon_t \right]_{a_0}^t$$

↑
when $x = a_0 \rightsquigarrow 0$ because $u_\varepsilon(\varepsilon, a_0) = 0$

It remains only $x = t$:

$$L_p(t, u(\varepsilon(t), t), u_x(\varepsilon(t), t)) \underbrace{u_\varepsilon(\varepsilon(t), t) \varepsilon_t(t)}_{L_p(t, v(t), p(t, v(t))) (\dot{v} - p^\downarrow(t, v(t)))}$$

Recall that

$$\begin{aligned} u(\varepsilon(t), t) &= v(t) \\ u_\varepsilon \cdot \varepsilon_t + u_x &= \dot{v} \quad \rightsquigarrow \quad u_\varepsilon \cdot \varepsilon_t = \dot{v} - u_x \\ &= \dot{v} - p(t, v(t)) \end{aligned}$$

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