

A.A. 2016/2017
Introduction to Calculus of Variations
Unabridged printout of lectures

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Calculus of Variations — LECTURE 01

Note Title

28/09/2016

Program

- ① Indirect method
- ② Direct method
- ③ Relaxation
- ④ Gamma - convergence

First part: give an idea of ① - ④
 Second part: more details and final examples

so final examples.

— o — o —

What is calculus of variations? Minimum pbms

$$\min \{ f(x) : x \in X \} \quad f: X \rightarrow \mathbb{R}$$

$\uparrow$ minimum value $\uparrow$ any set

$$\begin{aligned} \text{Minimum points} &= \{ x \in X : f(x) \text{ attains the minimum} \} \\ &= \operatorname{argmin} \{ f(x) : x \in X \} \subseteq X \end{aligned}$$

"Discussing" a minimum pbm means

- ① Proving or disproving that min exists
- ② If it exists, finding the minimum value and argmin
- ③ If it does not, understand why.

— o — o —

Trivial example $\min \{ \underbrace{x^2 + 5x}_{f(x)} : x \in \mathbb{R} \}$

Indirect method: $f'(x) = 2x + 5$, $f'(x) = 0 \Leftrightarrow x = -\frac{5}{2}$
 guess: the min. is achieved when $x = -\frac{5}{2}$ and
 $f(-\frac{5}{2}) = \text{min. value} = \frac{25}{4} - \frac{25}{2} = -\frac{25}{4}$

Rigorous proof: want to prove that $f(x) \geq -\frac{25}{4} \quad \forall x \in \mathbb{R}$

$$x^2 + 5x + \frac{25}{4} \geq 0 \quad \forall x \in \mathbb{R} \quad \left(x + \frac{5}{2}\right)^2 \geq 0$$

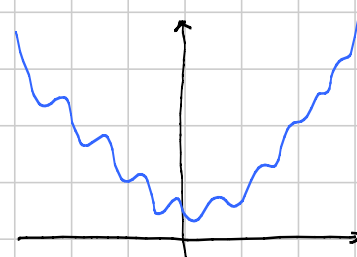
The last ineq. is true for every $x \in \mathbb{R}$, with equality $\Leftrightarrow x = -\frac{5}{2}$

Direct method: proving existence by means of general results

Consider $f(x)$. Observe that $\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = +\infty$

and that f is continuous in $\mathbb{R}$.

This guarantees the existence of the minimum.



Relaxation $\min \{ \underbrace{(x^2-3)^2}_{f(x)} : x \in \mathbb{Q} \}$

$f(x) \geq 0 \quad \forall x \in \mathbb{Q}$, but $f(x) = 0$ has no solution in $\mathbb{Q}$ and actually

$$\inf \{ (x^2-3)^2 : x \in \mathbb{Q} \} = 0$$

and the "minimizing sequences" are those converging to $\pm\sqrt{3}$.

When min does not exist, then the goal is

→ finding $\inf$

→ characterizing sequences $\{x_n\}$ with $f(x_n) \rightarrow \inf$.

Gamma convergence. Sequence of min probs depending on a parameter $n \in \mathbb{N}$, for example

$$M_n = \min \{ \underbrace{x^2 - \sin x + x^{2n}}_{f_n(x)} : x \in \mathbb{R} \}$$

Questions :

1 - $M_n \rightarrow ?$ as $n \rightarrow +\infty$

2 - what is the limit of minimum points ?

Idea: $M_n \rightarrow \min \{x^2 - \sin x : x \in [-1, 1]\}$

We will say that

$$f_n(x) \xrightarrow{\text{Gamma}} f_\infty(x) = \begin{cases} x^2 - \sin x & \text{if } x \in [-1, 1] \\ +\infty & \text{otherwise} \end{cases}$$

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INTEGRAL FUNCTIONALS

X = space of functions, for example

$$X = \{u \in C^3([0, 2]) : u(0) = 7\}$$

$F: X \rightarrow \mathbb{R}$ is usually called functional
(function defined in a space of functions)

For example

$$F(u) = \int_0^2 \{[u'(x)]^2 + 3 \cos(u(x))\} dx$$

↑
integral functional

We consider integral functionals of the following form

$$F(u) = \int_a^b L(x, u(x), u'(x)) dx$$

$L(x, s, p)$

where $u: [a, b] \rightarrow \mathbb{R}$
and
 $L: [a, b] \times \mathbb{R} \times \mathbb{R}$
 $\begin{matrix} x & s & p \\ \in & \in & \in \end{matrix}$

More general examples

$$F(u) = \int_a^b L(x, u', u'', \dots, u^{(n)}) dx$$

$$F(u, v) = \int_a^b L(x, u, v, u', v') dx$$

$$F(u) = \int_a^b L(x, u, u') dx \quad u: [a, b] \rightarrow \mathbb{R}^2$$

$$F(u) = \int_{\Omega} L(x, u(x), \nabla u(x)) dx \quad \begin{array}{l} u: \Omega \rightarrow \mathbb{R} \\ \text{but } \Omega \subseteq \mathbb{R}^m \end{array}$$

This course: $u: [a, b] \rightarrow \mathbb{R}$ and sometimes $u: [a, b] \rightarrow \mathbb{R}^k$

In one lecture $u: \Omega \rightarrow \mathbb{R}$ with $\Omega \subseteq \mathbb{R}^m$

More difficult: case where $u: \Omega \rightarrow \mathbb{R}^k$, with $n > 1, k > 1$

Which classes of functions $u: [a, b] \rightarrow \mathbb{R}$ do we consider?

Example $F(u) = \int_0^1 [\dot{u}^2 - xu] dx$

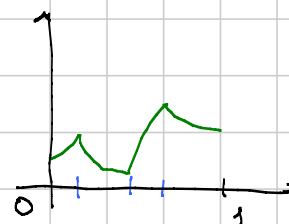
$$\min \{ F(u) : u \in C^1([0, 1]) \}$$

$$\min \{ F(u) : u \in C^{1*}([0, 1]) \} \quad \text{we ask more regularity}$$

We can ask less regularity, for example $u \in \text{PWC}^1([0, 1])$

Def. $u \in \text{PWC}^1([0, 1])$ if there exists $0 = x_0 < x_1 < \dots < x_n = 1$ such that

- $u \in C^1([x_{k-1}, x_k])$ for every $k = 1, \dots, n$
- $u \in C^0([0, 1])$



We can interpret $F(u)$ as

$$F(u) = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} [u^2 - xu] dx$$

Quite general fact Let us consider $F(u) = \int_a^b L(x, u, u') dx$

with $L: [a, b] \times \mathbb{R} \times \mathbb{R}$ continuous. Then

$$\inf \{ F(u) : u \in C^k([a, b]) \} \stackrel{\geq}{=} \inf \{ F(u) : u \in C^1([a, b]) \} \\ \stackrel{\geq}{=} \inf \{ F(u) : u \in \text{PWC}^1([a, b]) \}$$

[Proof : we can approximate every $u \in \text{PWC}^1([a, b])$
with a sequence $u_m \rightarrow u$ in such a way that

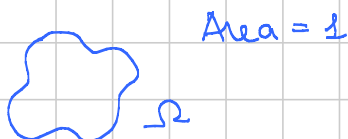
$$F(u_m) \rightarrow F(u)$$

We can also take $u_n \in C^\infty([a, b])$.]

Problems we can NOT consider $\leadsto$ many...

in particular geometric probs, namely problems in which the unknown is a set

Example $\min \{ \text{Length}(\partial\Omega) : \Omega \subseteq \mathbb{R}^2 \text{ is a set} \\ \text{with } \text{meas}(\Omega) = 1 \}$



$F = \boxed{\text{Length}}$ of boundary
 $\uparrow$ to be defined

$X =$ set of subsets of $\mathbb{R}^2$ with given Area

Calculus of Variations

LECTURE 02

Note Title

28/09/2016

INDIRECT METHOD

General setting X set $f: X \rightarrow \mathbb{R}$ function
 $\min \{f(x) : x \in X\}$

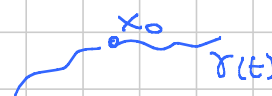
Let $x_0 \in X$ be any point. Let $\delta > 0$, and let $\gamma: (-\delta, \delta) \rightarrow X$ be curve such that $\gamma(0) = x_0$.

I can consider

$$\psi: (-\delta, \delta) \rightarrow \mathbb{R}$$

defined as

$$\psi(t) = f(\gamma(t)) \quad \forall t \in (-\delta, \delta)$$



Trivial fact Assume that $x_0 \in \arg\min \{f(x) : x \in X\}$.

Then $t=0$ is a minimum point for $\psi(t)$.

In particular

$$\psi'(0) = 0 \quad \text{if it exists}$$

Def. We define the FIRST VARIATION of f in the point x_0 wrt the curve γ as $\psi'(0)$.

$$\delta f(x_0, \gamma) := \psi'(0) \quad (\text{provided it exists})$$

Theorem (Necessary condition) If x_0 is a min. point, then

$$\delta f(x_0, \gamma) = 0 \quad \text{for every curve } \gamma: (-\delta, \delta) \rightarrow X \\ \text{with } \gamma(0) = x_0$$

— o — o —

Special case 1 $X = \text{vector space } V$

Take $\gamma(t) := x_0 + t v$
 $\uparrow$
 $v \in V$



$\delta f(x_0, v)$ = derivative of $f(x_0 + t v)$ computed at $t = 0$
 $\uparrow$
 direction

$$= \lim_{t \rightarrow 0} \frac{f(x_0 + t v) - f(x_0)}{t}$$

= Derivative of $f(x)$ in the direction v

Special case 2 $X = \text{affine space with reference space } V$

Once again if $x_0 \in X$ and $v \in V$ we have that $x_0 + t v \in X$ and so we can define

$$\delta f(x_0, v)$$

$\uparrow$ $\uparrow$
 $\in X$ $\in V$

as before

Examples $X = C^1([a, b]) = \text{vector space}$

$$X = \{ u \in C^1([a, b]) : u(a) = A \}$$

is an affine space with reference space

$$V = \{ u \in C^1([a, b]) : u(a) = 0 \}$$

— 0 — 0 —

Three main examples

$$F_1(u) = \int_a^b \dot{u}^2 dx, \quad F_2(u) = \int_a^b |\dot{u}| dx, \quad F_3(u) = \int_a^b |\dot{u}|^{1/2} dx$$

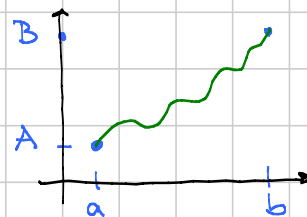
$$\min \begin{cases} F_1(u) \\ F_2(u) \\ F_3(u) \end{cases} : u \in C^1([a,b]), \quad u(a) = A, \quad u(b) = B$$

↑ given numbers ↑

Let us consider $F_1(u)$.

X is an affine space based on

$$V = \{ u \in C^1 \dots : u(a) = u(b) = 0 \}$$



Let u be a minimum point. Let $v \in V$ be any direction

$$\begin{aligned} \psi(t) &:= F(u+tv) = \int_a^b (\dot{u} + t\dot{v})^2 dx \\ &= \int_a^b \dot{u}^2 + 2t\dot{u}\dot{v} + t^2\dot{v}^2 dx \\ &= \int_a^b \dot{u}^2 + 2t \int_a^b \dot{u}\dot{v} + t^2 \int_a^b \dot{v}^2 \end{aligned}$$

$$\delta F_1(u, v) = \psi'(0) = 2 \int_a^b \dot{u}\dot{v}$$

If u is a minimum point, then

$$\int_a^b \dot{u}\dot{v} = 0 \quad \forall v \in V$$

EULER Equ, 1st form

Integrating by parts we obtain that

$$\begin{aligned} \int_a^b \dot{u}\dot{v} dx &= [\dot{u}(x)v(x)]_a^b - \int_a^b \ddot{u}(x)v(x) dx \\ &= \underbrace{\dot{u}(b)}_0 v(b) - \underbrace{\dot{u}(a)}_0 v(a) - \int_a^b \ddot{u} v dx \end{aligned}$$

↑ derivative ↑ integrate

If u is a min point AND $u \in C^2([0,1])$
then necessarily

$$\int_a^b \ddot{u} v \, dx = 0 \quad \forall v \in V$$

EULER EQN, 2nd integral form

Seems reasonable to guess that this is possible only if $\ddot{u} = 0$

$$\ddot{u}(x) = 0 \quad \forall x \in [a,b]$$

EULER EQN
in differential form

The solutions of our pbm is probably the unique function of the form

$$u(x) = mx + n$$

satisfying the BC (boundary conditions).

Formal proof that $u_0(x) = mx + n$, with suitable values of m and n , is the unique min. point for F_1 in X .

Take any $u \in X$. We can write $u = u_0 + v$ for a suitable $v \in V$. Then as before

$$\begin{aligned} F_1(u) &= \int_a^b (\ddot{u}_0 + \ddot{v})^2 \, dx = \int_a^b \ddot{u}_0^2 \, dx + 2 \int_a^b \ddot{u}_0 \ddot{v} \, dx + \int_a^b \ddot{v}^2 \, dx \\ &\quad \underbrace{\int_a^b \ddot{u}_0^2 \, dx}_{F_1(u_0)} \quad \underbrace{2 \int_a^b \ddot{u}_0 \ddot{v} \, dx}_{\begin{array}{l} \parallel \leftarrow u_0 \in C^2 \text{ and } v \equiv 0 \\ \text{at } a \text{ and } b \\ = -2 \int_a^b \ddot{u}_0 v \, dx \\ \parallel \\ 0 \end{array}} \\ &= F_1(u_0) + \int_a^b \ddot{v}^2 \, dx \end{aligned}$$

$$\begin{aligned} &\geq F_1(u_0) \\ \text{with equality} &\Leftrightarrow \int_a^b \ddot{v}^2 \, dx = 0 \Leftrightarrow \ddot{v}(x) \equiv 0 \Leftrightarrow v(x) \equiv 0 \\ &\quad \quad \quad \uparrow \\ &\quad \quad \quad v(a) = v(b) = 0 \end{aligned}$$

In the case of F_2 , u_0 is a minimum point again, but it is not unique.

Take any $u \in X$. Let us assume wlog that $B \geq A$

$$F_2(u) = \int_a^b |\dot{u}(x)| dx \geq \int_a^b \dot{u}(x) dx = u(b) - u(a) = B - A$$

$\stackrel{''}{=} F(u_0)$

BUT any nondecreasing function u is again a minimum point for F_2 because

$$F_2(u) = \int_a^b |\dot{u}(x)| dx = \int_a^b \dot{u}(x) dx = B - A = F(u_0)$$

↑
because $\dot{u}(x) \geq 0$ for every $x \in [a, b]$

This is $\Leftrightarrow$ because if $\dot{u}(x) < 0$ somewhere, then we have strict inequality.
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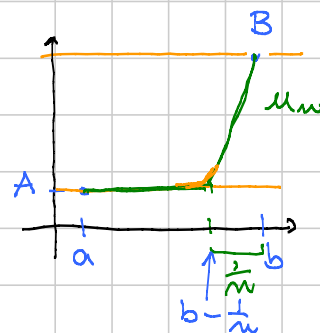
In the case of $F_3(u)$, the minimum pbm. has no solution unless $A = B$.

I claim that

$$\inf \{ F_3(u) : u \in C^1([a, b]) + B.C. \} = 0$$

This proves the the min does NOT exist

$$\begin{aligned} F_3(u_n) &= \int_a^b |\dot{u}_n(x)|^{1/2} dx \\ &= \int_{b-\frac{1}{n}}^b |\dot{u}_n(x)|^{1/2} dx \\ &= \int_{b-\frac{1}{n}}^b |n(B-A)|^{1/2} dx \\ &= |B-A|^{1/2} \sqrt{n} \frac{1}{n} = \frac{|B-A|^{1/2}}{\sqrt{n}} \rightarrow 0 \end{aligned}$$



What is important is the convexity of L with resp. to p

Calculus of Variations - LECTURE 03

Note Title

30/09/2016

FUNDAMENTAL LEMMA IN THE CALC. VAR.Main idea: suppose $f: [a,b] \rightarrow \mathbb{R}$ (continuous)

Assume that

$$\int_a^b f(x) v(x) dx = 0 \quad \forall v \in \text{some class of fcts.}$$

Then, under suitable assumptions, $f(x) = 0$ for every $x \in [a,b]$.Def. We say that $v \in C_c^\infty(a,b)$ if $\uparrow$
compact support $v \in C^\infty$ in (a,b) and there exists $[c,d] \subseteq (a,b)$ such that

$$v(x) = 0 \quad \forall x \notin [c,d]$$

 $(v \equiv 0 \text{ in a neighborhood of } \partial[a,b])$ **LEMMA** (FLCV) Suppose $f: [a,b] \rightarrow \mathbb{R}$ continuous such that

$$\int_a^b f(x) v(x) dx = 0 \quad \forall v \in C_c^\infty(a,b)$$

Then $f(x) = 0$ for every $x \in [a,b]$.

Two strategies

① by contradiction

② by density of C_c^∞ in C^0 .

Proof 1 Assume that $f \not\equiv 0$. Then there exists $x_0 \in [a, b]$ such that $f(x_0) \neq 0$.

WLOG I can assume $x_0 \in (a, b)$ and $f(x_0) > 0$.

Then there exist $\delta > 0$ and $r > 0$

such that

$$[x_0 - r, x_0 + r] \subseteq (a, b)$$

$$f(x) \geq \delta \quad \forall x \in [x_0 - r, x_0 + r]$$

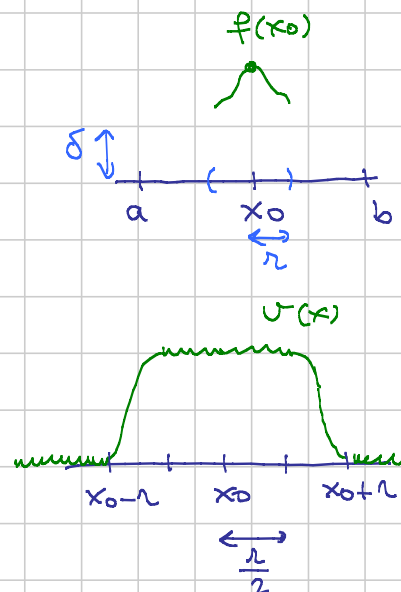
Standard exercise: there exists

$\psi \in C_c^\infty(a, b)$ such that

- $\psi(x) = 1$ on $[x_0 - \frac{r}{2}, x_0 + \frac{r}{2}]$
- $\psi(x) = 0$ outside $[x_0 - r, x_0 + r]$
- $0 \leq \psi(x) \leq 1 \quad \forall x \in \mathbb{R}$

Now

$$\begin{aligned} \int_a^b f(x) \psi(x) dx &= \int_{x_0-r}^{x_0+r} f(x) \psi(x) dx \\ &\geq \int_{x_0-\frac{r}{2}}^{x_0+\frac{r}{2}} f(x) \psi(x) dx \\ &= \int_{x_0-\frac{r}{2}}^{x_0+\frac{r}{2}} \underbrace{f(x)}_{\geq \delta} dx \geq r\delta > 0 \end{aligned}$$



Proof 2 General fact: for every continuous function

$g : [a, b] \rightarrow \mathbb{R}$ there exists a sequence of functions

$g_n \in C_c^\infty(a, b)$ such that

(i) $\exists M \in \mathbb{R}$ s.t. $|g_n(x)| \leq M$ for every $x \in [a, b]$, $n \in \mathbb{N}$

(ii) $g_n \rightarrow g$ uniformly on compact subsets of (a, b)

Use this fact with $g = f$ and use $\psi(x) \equiv g_n(x)$

$C_c^\infty(a, b)$

$g_n \rightarrow f$

$$0 = \int_a^b f(x) g_n(x) dx \longrightarrow \int_a^b f(x)^2 dx$$

↑
because of
(i) and (ii)

Since $f(x)$ is continuous, this is possible only if $f \equiv 0$.

Du Bois - REYMOND LEMMA

Assume $f: [a, b] \rightarrow \mathbb{R}$ cont.

Assume

$$(*) \quad \int_a^b f(x) v(x) dx = 0 \quad \forall v \in C_c^\infty(a, b) \text{ s.t. } \int_a^b v(x) dx = 0$$

Then $f(x)$ is constant in $[a, b]$.

Brutal proof Assume we can integrate by parts

$$0 = \int f v = - \int \dot{f} V \quad \text{where } V'(x) = v(x) \text{ in } (a, b) \text{ and } V(a) = V(b) = 0$$

Then FLCV $\Rightarrow \dot{f} \equiv 0 \Rightarrow f$ is constant.

True proof. If f satisfies $(*)$, then $f + c$ satisfies $(*)$ for every constant c .

Let us assume that f is not constant. Then there exist x_0 and y_0 in (a, b) such that $f(x_0) \neq f(y_0)$.

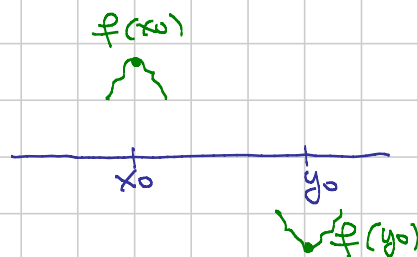
Up to choosing the right constant c , I can assume that

$$f(x_0) = -f(y_0)$$

Choose $r > 0$ and $\delta > 0$ s.t.

$$f(x) \geq \delta \text{ in } [x_0 - r, x_0 + r]$$

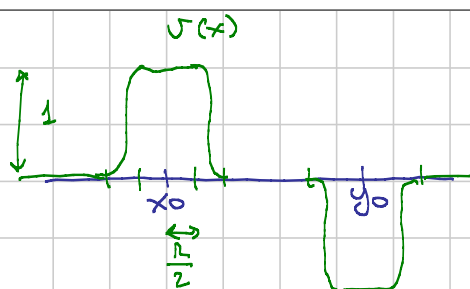
$$f(x) \leq -\delta \text{ in } [y_0 - r, y_0 + r]$$



Finally, choose v this way $\leadsto$

Now $\int_a^b v(x) dx = 0$, but

$$\int_a^b f(x) v(x) dx > 0.$$



Remark I can also provide a second proof as in the case of FLCV. Now I need an approx. with extra requirement that

$$\int g(x) dx = \int g_n(x) dx = 0.$$

Remark Let $f: [a, b] \rightarrow \mathbb{R}$ be given. Let us consider

$$V = \left\{ v: [a, b] \rightarrow \mathbb{R} : v \text{ is cont. and } \int_a^b f(x) v(x) dx = 0 \right\}$$

Then V has the following properties

- (1) V is a vector space
- (2) V is closed under uniform convergence, namely if $v_n \rightarrow v_\infty$ unif. and $v_n \in V \Rightarrow v_\infty \in V$
- (3) V is closed under unif. conv. in compact subsets of $[a, b] \setminus S$ (where S is a finite set) provided that $v_n \rightarrow v_\infty$ with v_n equibounded.

Consequence In order to conclude that $f \equiv 0$ in $[a, b]$ it is enough to check that

$$\int_a^b f(x) v(x) dx = 0 \quad \forall v \in F$$

better: the closure of $\text{span} F$

where F is a set of functions whose closure $\uparrow$ with respect to (3) is dense in $C^0([a, b])$ (if $\text{span} F$ is dense in C_c^∞ , then it is dense in C^0).

Example 1 $\min \left\{ \underbrace{\int_a^b \dot{u}^2 dx}_{F(u)} : \begin{array}{l} u \in C^1([a,b]) \\ u(a) = A \quad u(b) = B \\ \int_a^b u(x) dx = 8 \end{array} \right\}$

X

The space X is affine with reference vector space

$$V = \left\{ v \in C^1 : v(a) = v(b) = \int_a^b v(x) dx = 0 \right\}$$

$$\psi(t) := F(u + tv) = F(u) + 2t \int_a^b \dot{u} \dot{v} dx + t^2 \int_a^b \dot{v}^2 dx$$

$$\delta F(u, v) = \psi'(0) = 2 \int_a^b \dot{u} \dot{v} dx \quad \forall v \in V$$

Euler eqn
1st form.

Integrating by parts we obtain

$$\int_a^b \ddot{u} v dx = 0 \quad \forall v \in V$$

$$\text{DBR Lemma} \Rightarrow \ddot{u} = \text{constant} \Rightarrow u(x) = ax^2 + bx + c$$

Easy to see that $BC + \int u = 8$ allow to determine a, b, c in a unique way.

In the usual way one can prove that u is actually a minimum point.

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Calculus of Variations — LECTURE 04

Note Title

30/09/2016

BIRTH OF BOUNDARY CONDITIONS

$$F(u) = \int_a^b (\dot{u}^2 + u^2 + \tau u) dx$$

Example 1 $X_1 = \{ u \in C^1([a,b]) : u(a) = A, u(b) = B \}$

$$V_1 = \{ v \in C^1([a,b]) : v(a) = v(b) = 0 \}$$

$$\begin{aligned} \psi(t) = F_1(u+tv) &= \int_a^b (\dot{u} + t\dot{v})^2 + (u+tv)^2 + \tau(u+tv) \\ &= \int_a^b (\dot{u}^2 + u^2 + \tau u) + t \int_a^b (2\dot{u}\dot{v} + 2uv + \tau v) + t^2 \int_a^b \dot{v}^2 + v^2 \end{aligned}$$

$$\psi'_1(0) = \int_a^b (2\dot{u}\dot{v} + 2uv + \tau v) dx$$

If u is a min point, then $\underbrace{\int_a^b (2\dot{u}\dot{v} + 2uv + \tau v) dx}_{\delta F_1(u,v)} = 0 \quad \forall v \in V_1$

Euler eqn
1st form

Integr. by parts:

$$\underbrace{\int_a^b (-2\ddot{u} + 2u + \tau) v dx}_{2^{\text{nd}} \text{ form of EULER Eqn}} = 0 \quad \forall v \in V_1$$

$$\text{FLCV} \Rightarrow 2\ddot{u} = 2u + \tau$$

Euler Eqn in Diff. Form

$$u(x) = c_1 e^x + c_2 e^{-x} - \frac{\tau}{2} \quad \leftarrow \text{general solution}$$

Imposing the BC $u(a) = A, u(b) = B$ we can find c_1 and c_2 in a unique way.

$$\begin{cases} 2\ddot{u} = 2u + f \\ u(a) = A \\ u(b) = B \end{cases} \quad \begin{array}{l} \text{DIRICHLET} \\ \text{boundary conds.} \end{array} \quad (\text{DBC})$$

Example 2 $X_2 = \{u \in C^1([a,b]) : u(a) = A\}$

$$V_2 = \{v \in C^1 \dots : v(a) = 0\}$$

As before we compute $\psi_2(t)$ and $\psi_2'(0)$, and we find

$$\int_a^b (2\ddot{u}v + 2uv + fv) = 0 \quad \forall v \in V_2 \quad \text{EU-E-1}$$

What happens when we integrate by parts?

$$0 = \int_a^b (-2\ddot{u} + 2u + f)v dx + [2\dot{u}v]_a^b \quad \forall v \in V_2$$

" $2\dot{u}(b)v(b) - 2\dot{u}(a)v(a)$

$\underbrace{\quad}_{=0}$

Step 1 We limit ourselves to function $v \in C_c^\infty(a,b) \subseteq V_2$

We get $\int_a^b (-2\ddot{u} + 2u + f)v dx = 0 \quad \forall v \in C_c^\infty(a,b)$

FLCV $\Rightarrow 2\ddot{u} = 2u + f$ Diff. form. of Euler equ.

Step 2 Let's go back to 2nd form of EE (Euler-Lagrange)
We have that

$$0 = 2\dot{u}(b)v(b) \quad \forall v \in V_2$$

This means that $\dot{u}(b) = 0$.

We ended up with

$$\begin{cases} 2\ddot{u} = 2u + 7 \\ u(a) = A \quad \text{DBC} \\ \dot{u}(b) = 0 \quad \text{NBC (NEUMANN BC)} \end{cases}$$

Example 3 $X_3 = \{u \in C^1([a,b])\} = V_3$

As before, we find that

$$0 = \int_a^b (-2\ddot{u} + 2u + 7)v \, dx + \dot{u}(b)v(b) - \dot{u}(a)v(a) \quad \forall v \in V_3$$

E-E-2nd form.

Step 1 $V_3 \subseteq C_c^\infty(a,b)$ $\text{FLCV} \Rightarrow 2\ddot{u} = 2u + 7$

Step 2 Go back to 2nd form $\dot{u}(b)v(b) - \dot{u}(a)v(a) = 0$
 $\forall v \in V_3$

- Take $v \in V_3$ with $v(b) = 1$ and $v(a) = 0 \leadsto \dot{u}(b) = 0$
- Take " $v(b) = 0$ " $v(a) = 1 \leadsto \dot{u}(a) = 0$

$$\begin{cases} 2\ddot{u} = 2u + 7 \\ \dot{u}(a) = 0 \quad \text{NBC} \\ \dot{u}(b) = 0 \quad \text{NBC} \end{cases}$$

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Example 4 $X_4 = \{u \in C^1([a,b]) : u(a) = u(b)\} = V_4$

As before we obtain the second form of EE $\forall v \in V_4$

As in step 1 we deduce $2\ddot{u} = 2u + 7$

And in step 2 we obtain

$$\dot{u}(b)v(b) - \dot{u}(a)v(a) = 0 \quad \forall v \in V_4$$

$\overset{k}{\dot{u}(b)} - \overset{k}{\dot{u}(a)}$

$$k(\dot{u}(b) - \dot{u}(a)) = 0$$

Since we can take $k \neq 0$, we end up with

$$\begin{cases} 2\ddot{u} = 2u + 7 \\ u(a) = u(b) \\ \dot{u}(a) = \dot{u}(b) \end{cases} \quad \begin{array}{l} \text{Periodic boundary} \\ \text{conditions} \end{array} \quad (\text{PBC})$$

Example 5

$$F(u) = \int_a^b (\dot{u}^2 + 7\dot{u} + u^4) dx$$

$$X = \{u \in C^1 : u(b) = B\}$$

$$\psi(t) = F(u + tv) = F(u) + t \int_a^b (2\dot{u}\dot{v} + 7\dot{v} + 4u^3v) dx + o(t)$$

$$\delta F(u, v) = \int_a^b (2\dot{u}\dot{v} + 7\dot{v} + 4u^3v) dx \quad [=0]$$

Integration by parts

$$\begin{aligned} \int_a^b (2\dot{u} + 7)\dot{v} &= [(2\dot{u} + 7)v]_a^b - \int_a^b 2\ddot{u}v \\ &= 2\dot{u}(b)v(b) + 7v(b) - 2\dot{u}(a)v(a) - 7v(a) \end{aligned}$$

Second form of Euler equ.

$$\int_a^b (-2\ddot{u} + 4u^3)v dx - (2\dot{u}(a) + 7)v(a) = 0$$

Step 1 $v \in C_c^\infty \dots$ FLCV ... $\ddot{u} = 2u^3$

Step 2 Take $v(a) = 1$ and deduce $\dot{u}(a) = -\frac{7}{2}$

$$\begin{cases} \ddot{u} = 2u^3 \\ u(b) = B \\ \dot{u}(a) = -\frac{7}{2} \end{cases} \quad \begin{array}{l} \text{EE} \\ \text{DBC from the beginning} \\ \text{NBC born on the road} \end{array}$$

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Example 6 $F(u) = \int_a^b (\ddot{u}^2 + u^3) dx$

$$X = \{u \in C^2 \dots : u(a) = A, \dot{u}(b) = B\}$$

$$\psi(t) = F(u) + t \int_a^b (2\ddot{u}\ddot{v} + 3u^2v) + o(t)$$

$$\delta F(u, v) = \int_a^b (2\ddot{u}\ddot{v} + 3u^2v) dx$$

$$\begin{aligned} \int \ddot{u}\ddot{v} &= [\ddot{u}\dot{v}]_a^b - \int_a^b u''' \dot{v} \\ &= [\ddot{u}\dot{v}]_a^b - [u'''v]_a^b + \int_a^b u''''v \end{aligned}$$

Second form of Euler equ

$$\int_a^b [2u'''' + 3u^2]v dx + 2[\ddot{u}\dot{v}]_a^b - 2[u'''v]_a^b$$

Step 1 $\rightsquigarrow 2u'''' = -3u^2$
 $u(a) = A$
 $\dot{u}(b) = B$

We need two more condition

Step 2 $\rightsquigarrow \underbrace{\ddot{u}(b)\dot{v}(b)}_{=0} - \ddot{u}(a)\dot{v}(a) - \underbrace{u'''(b)v(b)}_{=0} + u'''(a)\underbrace{v(a)}_{=0} = 0$

Take $v \in V$ with $\dot{v}(a) = 0$ and $v(b) = 1 \rightsquigarrow u'''(b) = 0$
 $\dot{v}(a) = 1$ and $v(b) = 0 \rightsquigarrow \ddot{u}(a) = 0$

$\ddot{u}(a) = 0$
 $u'''(b) = 0$ } Two extra conditions born on the road.

Calculus of Variations — LECTURE 05

Note Title

05/10/2016

DIFFERENT FORMS OF EULER Equation↑
(AKA EULER-LAGRANGE Equ)

$$F(u) = \int_a^b L(x, u(x), u'(x)) dx$$

↑
Lagrangian

$$L: [a, b] \times \mathbb{R} \times \mathbb{R}$$

Let us assume that u is a minimum point of F subject to some BC, for example DBC

$$u(a) = A, \quad u(b) = B$$

Then $0 = \delta F(u, v) = \lim_{t \rightarrow 0} \frac{F(u+tv) - F(u)}{t} = \psi'(0)$

$$\psi(t) := F(u+tv).$$

General result Let $f: \underbrace{[a, b]}_x \times \underbrace{[-\delta, \delta]}_t \rightarrow \mathbb{R}$

Let us set

$$\psi(t) := \int_a^b f(x, t) dx \quad t \in [-\delta, \delta]$$

(1) If $f(x, t)$ is continuous in $[a, b] \times [-\delta, \delta]$, then $\psi(t)$ is continuous in $[-\delta, \delta]$

(2) If $f_t(x, t)$ is continuous in $[a, b] \times [-\delta, \delta]$, then $\psi'(t)$ exists and it is given by

$$\psi'(t) = \int_a^b f_t(x, t) dx$$

(the derivative of the integral is the integral of the derivative)

Let us take

$$\psi(t) = F(u+tv) = \int_a^b \overbrace{L(x, u(x)+tv(x), \dot{u}(x)+t\dot{v}(x))}^{f(x,t)} dx$$

$$\psi'(t) = \int_a^b \frac{d}{dt} L(x, u+tv, \dot{u}+t\dot{v}) dx$$

CHAIN RULE $\rightarrow$
$$= \int_a^b L_s(x, u+tv, \dot{u}+t\dot{v})v + L_p(x, u+tv, \dot{u}+t\dot{v})\dot{v} dx$$

When $t=0$ we get

$$\psi'(0) = \delta F(u, v) = \int_a^b L_s(x, u(x), \dot{u}(x))v(x) + L_p(x, u(x), \dot{u}(x))\dot{v}(x) dx$$

$$\int_a^b [L_s(x, u, \dot{u})v + L_p(x, u, \dot{u})\dot{v}] dx = 0$$

First integral form of ELE. Sufficient conditions:

- L is of class C^1 ,
- u is of class C^1 .

Let us assume that v vanishes at the boundary. If we integrate by parts we obtain that

$$\int_a^b \left[L_s(x, u, \dot{u}) - \frac{d}{dx} L_p(x, u, \dot{u}) \right] v(x) dx = 0$$

Second integral form of ELE. Sufficient conditions

- L is of class C^2
- u is of class C^2

Exploiting FLCV we finally obtain

$$\frac{d}{dx} L_p(x, u, \dot{u}) = L_s(x, u, \dot{u})$$

ELE in standard
differential form
(second order diff. equ.)
ODE

Oss. The LHS can be rewritten as

$$L_{p_x}(x, u, \dot{u}) + L_{p_s}(x, u, \dot{u}) \dot{u} + L_{p_{pp}}(x, u, \dot{u}) \ddot{u} = L_s(x, u, \dot{u})$$

Neumann condition in the general setting.

$$\int_a^b L_s(x, u, \dot{u}) v + L_p(x, u, \dot{u}) \dot{v} = 0$$

$$\int_a^b L_s(x, u, \dot{u}) v - [L_p(x, u, \dot{u})]' v + [L_p(x, u, \dot{u}) v]_{x=a}^{x=b}$$

$$L_p(b, u(b), \dot{u}(b)) v(b) - L_p(a, u(a), \dot{u}(a)) v(a) = 0$$

NEUMANN CONDITION : $L_p(x, u(x), \dot{u}(x)) = 0$ in the endpoint
where the condition is needed.

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DU BOIS REYMOND ELE 1st integral form

$$\int_a^b L_s(x, u, \dot{u}) v + L_p(x, u, \dot{u}) \dot{v} = 0$$

↓ transform into $\dot{v}$

Let us introduce the antiderivative of $L_s(x, u, \dot{u})$, namely

$$\hat{L}(x) := \int_a^x L_s(y, u(y), \dot{u}(y)) dy \quad \forall x \in [a, b]$$

Integrating by parts we find (assume $v \equiv 0$ at boundary)

$$\int_a^b -\hat{L}(x) \dot{v} + L_p(x, u, \dot{u}) \dot{v} = 0 = \int_a^b [-\hat{L}(x) + L_p(x, u, \dot{u})] \dot{v} = 0$$

any C^0 function
with 0 average

because $\int_a^b \dot{v}(x) dx = v(b) - v(a) = 0$

DBR Lemma $\Rightarrow -\hat{L}(x) + L_p(x, u, \dot{u}) = \text{constant}$

$$L_p(x, u(x), \dot{u}(x)) = \text{const} + \int_a^x L_s(y, u(y), \dot{u}(y)) dy$$

1st order
ODE

ELE in DBR form. Does NOT require any SECOND derivative (just u and L of class C^1)

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ELE in ERDMANN FORM

We assume that the plan is AUTONOMOUS, namely L does NOT depend on x : $L(s, p)$

$$[L_p(u, \dot{u})]' \dot{u} = L_s(u, \dot{u}) \dot{u}$$

$$\text{LHS} = [L_p(u, \dot{u}) \dot{u}]' - L_p(u, \dot{u}) \ddot{u} = L_s(u, \dot{u}) \dot{u}$$

$$[L_p(u, \dot{u}) \dot{u}]' = L_s(u, \dot{u}) \dot{u} + L_p(u, \dot{u}) \ddot{u} = [L(u, \dot{u})]'$$

It follows that

$$L_p(u, \dot{u}) \dot{u} - L(u, \dot{u}) = \text{constant}$$

ERDMANN FORM
(if L is autonomous)
1st order ODE

Remark Standard and Erdmann forms are not equivalent

- (1) If u satisfies ELE standard, then u satisfies ELE-ERDMANN
- (2) If u satisfies ELE-ERDMANN, then u satisfies ELE stand. for all points $x \in [a, b]$ s.t. $\dot{u}(x) \neq 0$.

Generalization 1 L depending on higher order derivatives

$$F(u) = \int_a^b L(x, u, \dot{u}, \ddot{u}, \dots, u^{(k)}) dx \quad L(x, s, p_1, \dots, p_k)$$

$\psi(t) := F(u + tv) \dots$ same computation $\dots$

$$\int_a^b L_s(\dots) v + \sum_{i=1}^k L_{p_i}(\dots) v^{(i)} = 0 \quad \text{First integral form}$$

$$\int_a^b L_s(\dots) v + \sum_{i=1}^k (-1)^i \frac{d^i}{dx^i} L_{p_i}(\dots) v = 0 \quad \text{Second integral form}$$

$$FLCV \Rightarrow \sum_{i=1}^k (-1)^{i+1} [L_{p_i}(\dots)]^{(i)} = L_s(\dots)$$

Generalization 2 F depends on more unknowns

$$F(u_1, \dots, u_k) = \int_a^b L(x, u_1, \dots, u_k, \dot{u}_1, \dots, \dot{u}_k) \\ L(x, s_1, \dots, s_k, p_1, \dots, p_k)$$

We use a vector of directions $(v_1, \dots, v_k)$

$$\psi(t) := F(u_1 + tv_1, \dots, u_k + tv_k)$$

$$\begin{aligned} \psi'(0) &= \int_a^b \sum_{i=1}^k [L_{s_i}(\dots) v_i + L_{p_i}(\dots) \dot{v}_i] \\ &= \int_a^b \sum_{i=1}^k [L_{s_i}(\dots) - [L_{p_i}(\dots)]'] v_i = 0 \end{aligned}$$

Use $v_1 = \text{any } C_c^\infty$ and $v_2 = \dots = v_k = 0$

$$[L_{p_1}(\dots)]' = L_{s_1}(\dots) \text{ and so on}$$

We finally obtain a system of the form

$$[L_{p_i}(x, u_1, \dots, u_k, \dot{u}_1, \dots, \dot{u}_k)]' = L_s(\dots) \quad i = 1, \dots, k$$

System of k ODEs of order 2.

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Calculus of Variations — LECTURE 06

Note Title

05/10/2016

Question: how to prove that a given function $u_0(x)$ is a/the minimum point of a functional $F(u)$

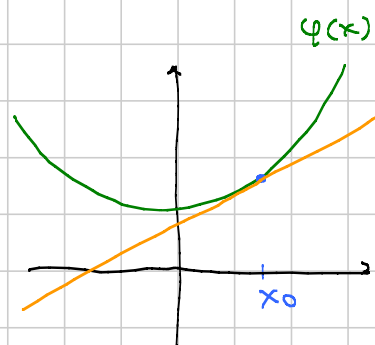
Basic fact about convex functions

Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a convex function.

Let $x_0 \in \mathbb{R}$.

Then there exists $m \in \mathbb{R}$ such that

$$\varphi(x) \geq \underbrace{\varphi(x_0) + m(x - x_0)}_{\text{line through } (x_0, \varphi(x_0))} \quad \leftarrow \text{strict inequality when } \varphi \text{ is strictly convex and } x \neq x_0$$



Proof when φ is of class C^2 Use Taylor with Lagrange Remainder

Define $x - x_0 = R$. Then

$$\begin{aligned} \varphi(x) &= \varphi(x_0 + R) = \varphi(x_0) + \varphi'(x_0)R + \frac{1}{2} \varphi''(x_0 + \theta R) R^2 \\ &\quad \uparrow \text{Taylor-Lagrange} \\ &\geq \varphi(x_0) + \underbrace{\varphi'(x_0)}_{m} (x - x_0) \end{aligned}$$

The result is true even if f is not of class C^1

[Proof. In every $x_0 \in \mathbb{R}$ one can define

$$f'_+(x_0) = \lim_{R \rightarrow 0^+} \frac{f(x_0 + R) - f(x_0)}{R}$$

$$f'_-(x_0) = \lim_{R \rightarrow 0^-} \frac{f(x_0 + R) - f(x_0)}{R}$$



For every $x_0 \in \mathbb{R}$ it turns out that $f'_-(x_0) \leq f'_+(x_0)$ and any m in between does the job.

The same is true in 2D or any dimension. For example in 2D if $f(x,y)$ is a convex function, then for every $(x_0, y_0) \in \mathbb{R}^2$ there exists a plane such that

$$f(x,y) \geq f(x_0, y_0) + m_1(x-x_0) + m_2(y-y_0)$$

with strict inequality if ---

MINIMALITY THROUGH CONVEXITY

$$F(u) = \int_a^b L(x, u(x), \dot{u}(x)) dx \quad L(x, s, p)$$

Want to minimize F in $C^1([a,b])$ subject to BC
 $u(a) = A, u(b) = B$

Let us assume that $u_0(x)$ is a solution to ELE with BC.

If $(s,p) \rightarrow L(x,s,p)$ is convex w.r.t (s,p) for every $x \in [a,b]$

Then $u_0(x)$ is a minimum point for $F(u)$

If the same function is strictly convex, then $u_0(x)$ is the unique minimum point.

Proof. Let us take any $w \in C^1$ with same BC.

Can write $w = u_0 + v$

$\uparrow$ class C^1 with zero BC

$$F(w) = F(u_0 + v) = \int_a^b L(x, u_0 + v, \dot{u}_0 + \dot{v}) dx$$

$$L \text{ is convex} \rightarrow \geq \int_a^b L(x, u_0, \dot{u}_0) dx$$

$\leftarrow F(u_0)$

$$+ \underbrace{\int_a^b L_s(x, u_0, \dot{u}_0) v + L_p(x, u_0, \dot{u}_0) \dot{v}}_{=0 \text{ because of ELE}}$$

This proves that $F(w) \geq F(u_0)$ for every admissible w .

Let us assume now that L is strictly convex and $\dot{v} \neq 0$ or $v \neq 0$. Then there exists at least one point $x_0 \in [a, b]$ where

$$L(x_0, u_0(x_0) + v(x_0), \dot{u}_0(x_0) + \dot{v}(x_0)) > L(x_0, u_0, \dot{u}_0) + L_s v + L_p \dot{v}$$

The same is true in a neighborhood (because of cont.), and therefore $F(w) > F(u_0)$

Basic example

$$F(u) = \int_a^b L(\dot{u}) dx \quad L(p)$$

$$\text{ELE} \quad [L_p(\cdot)]' = L_s \quad [L_p(\dot{u})]' = 0$$

$$L_{pp}(\dot{u}) \ddot{u} = 0$$

If $L_{pp}(p) > 0$ for every $p \in \mathbb{R}$, then $\ddot{u} = 0$, which means that the only solution is the line. This is also the unique minimizer.

More generally let us consider $\min \left\{ \int_a^b L(\dot{u}) : \text{DBC} \right\}$

(1) If L is convex (no regularity required), then the line is a minimizer

(2) If L is strictly convex, then it is the unique minimizer.

Proof. Let $u_0(x) = \alpha x + \beta$ be the line. Let w be any other admissible function. Write $w = u_0 + v$. Take w such that

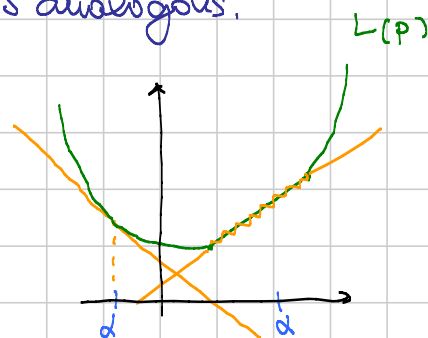
$$L(p) \geq L(\alpha) + w(p - \alpha)$$

As before

$$\begin{aligned}
 F(w) &= \int_a^b L(\dot{w} + \dot{v}) = \int_a^b L(\alpha + \dot{v}) \\
 &\geq \int_a^b L(\alpha) + w \dot{v} \\
 &= F(w_0) + w(v(b) - v(a)) = F(w_0)
 \end{aligned}$$

The argument for the strictly convex case is analogous.

Remark If L is convex (but not necessarily strictly convex) and α is such that $L(p)$ is always above the tangent line, then $w_0(x)$ is again the unique minimizer.



Second tool in order to prove minimality

TRIVIAL LEMMA Let X be a set. Let $x_0 \in X$.
Let $F: X \rightarrow \mathbb{R}$ and $G: X \rightarrow \mathbb{R}$

Let us assume that

(i) $F(x) \geq G(x) \quad \forall x \in X$

(ii) $F(x_0) = G(x_0)$

(iii) $G(x) \geq G(x_0) \quad \forall x \in X$ (namely $x_0 \in \operatorname{argmin} \{G(x) : x \in X\}$)

Then

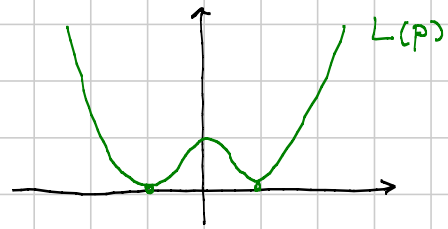
$$x_0 \in \operatorname{argmin} \{F(x) : x \in X\}.$$

Proof. $F(x) \underset{(i)}{\geq} G(x) \underset{(iii)}{\geq} G(x_0) \underset{(ii)}{=} F(x_0) \quad \forall x \in X. \quad \square$

Example

$$\min \left\{ \int_0^1 \underbrace{(\dot{u}^2 - 1)^2}_{L(\dot{u})} dx : u \in C^1, u(0)=2, u(1)=4 \right\}$$

$$L(p) = (p^2 - 1)^2$$



DOUBLE WELL POTENTIAL

Conjecture: the line $u_0(x) = 2 + 2x$ is the unique minimizer.

Let $F(u)$ be the given functional, and let

$$G(u) := \int_0^1 \hat{L}(\dot{u}) dx$$

where $\hat{L}(p)$ is the convexification of $L(p)$



$$F(u) \geq G(u) \quad (\text{because } L(p) \geq \hat{L}(p) \quad \forall p \in \mathbb{R})$$

$$F(u_0) = G(u_0) \quad (\text{because } L(2) = \hat{L}(2))$$

$$u_0 \text{ is a minimum point of } G(u) \quad (\text{because } \hat{L} \text{ is convex})$$

Trivial lemma $\Rightarrow u_0(x) = 2x + 2$ is a minimum point of F

Moreover, it is the unique minimum point of F because
 " " " " " " " " G , because
 $p = 2$ falls in a region where $\hat{L}$ is strictly convex.

Calculus of Variations - LECTURE 07

Note Title

07/10/2016

Point-to-curve probs

$$F(u) = \int_a^b L(x, u, \dot{u}) dx$$

Given a function $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ of class C^1

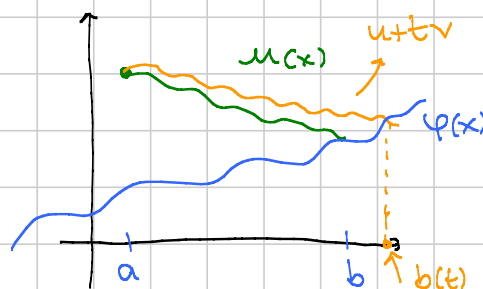
$$F(u, b) = \int_a^b L(x, u, \dot{u}) dx$$

Goal: minimizing $F(u, b)$ among all pairs (u, b) with

$$u \in C^1([a, b])$$

$$u(a) = A \leftarrow \text{given}$$

$$u(b) = \varphi(b)$$



As always, let us take any $v \in C^\infty([a, +\infty))$ with $v(a) = 0$. Consider

$$u(x) + t v(x)$$

Find whether and where the new function crosses $\varphi(x)$.

$$u(b(t)) + t v(b(t)) = \varphi(b(t))$$

$$b: (-\delta, \delta) \rightarrow \mathbb{R}$$

$$b(0) = b$$

$$u(b(t)) + t v(b(t)) - \varphi(b(t)) = 0$$

$$\Phi(x, t) := u(x) + t v(x) - \varphi(x)$$

$$\Phi(b(t), t) = 0$$

$\leadsto$ implicit function theorem $\leadsto$ can find $b(t)$ if

$$\Phi_x(b, 0) \neq 0$$

$$\Phi_x(x, t) = \dot{u}(x) + t \dot{v}(x) - \dot{\varphi}(x)$$

$$\Phi_x(b, 0) \neq 0 \Leftrightarrow \dot{u}(b) - \dot{\varphi}(b) \neq 0$$

Let us assume that $\ddot{u}(b) - \ddot{\varphi}(b) \neq 0$. Then $b(t)$ exists and

$$\dot{b}(0) = - \frac{\Phi_t(b, 0)}{\Phi_x(b, 0)} = - \frac{v(b)}{\ddot{u}(b) - \ddot{\varphi}(b)}$$

Let us set

$$\psi(t) := \int_a^{b(t)} L(x, u(x) + tv(x), \dot{u}(x) + t\dot{v}(x)) dx$$

$$\begin{aligned} \psi'(t) &= \dot{b}(t) L(b(t), u(b(t)) + tv(b(t)), \dot{u}(b(t)) + t\dot{v}(b(t))) \\ &\quad + \int_a^{b(t)} L_s(x, u+tv, \dot{u}+t\dot{v}) v + L_p(x, u+tv, \dot{u}+t\dot{v}) \dot{v} dx \end{aligned}$$

Setting $t=0$:

$$\begin{aligned} 0 = \psi'(0) &= \dot{b}(0) L(b, u(b), \dot{u}(b)) + \int_a^b L_s(x, u, \dot{u}) v + L_p(x, u, \dot{u}) \dot{v} \\ &= \int_a^b L_s(\dots) v - [L_p(\dots)]' v \\ &\quad + L_p(b, u(b), \dot{u}(b)) v(b) \end{aligned}$$

$$0 = \psi'(0) = - \frac{v(b)}{\ddot{u}(b) - \ddot{\varphi}(b)} L(b, u(b), \dot{u}(b)) + L_p(b, u(b), \dot{u}(b)) v(b)$$

Since we can take $v(b) = 1$, we obtain

$$L_p(b, u(b), \dot{u}(b)) [\ddot{u}(b) - \ddot{\varphi}(b)] = L(b, u(b), \dot{u}(b))$$

Boundary condition for point-to-curve problem
(TRANSVERSALITY)

In addition we have the usual ELE

$$[L_p(x, u(x), \dot{u}(x))]' = L_s(x, u(x), \dot{u}(x)) \quad \forall x \in [a, b]$$

The transversality condition holds true even if $\dot{u}(b) = \dot{\varphi}(b)$

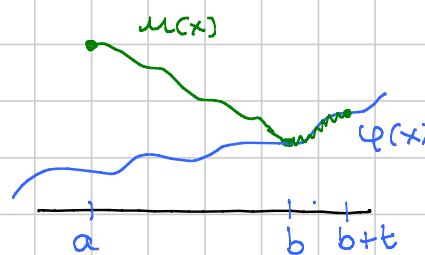
1st step Consider an "extension" of u up to $b+t$ obtained by following $\varphi(x)$

$$\psi(t) := \int_a^b L(x, u, \dot{u}) dx + \int_b^{b+t} L(x, \varphi, \dot{\varphi}) dx$$

$$\psi'(t) = L(b+t, \varphi(b+t), \dot{\varphi}(b+t))$$

$$\psi'(0) = L(b, \varphi(b), \dot{\varphi}(b)) = L(b, u(b), \dot{u}(b))$$

↑
because $\dot{u}(b) = \dot{\varphi}(b)$



If I could say that $\psi'(0) = 0$, then I could conclude that

$$L(b, u(b), \dot{u}(b)) = 0$$

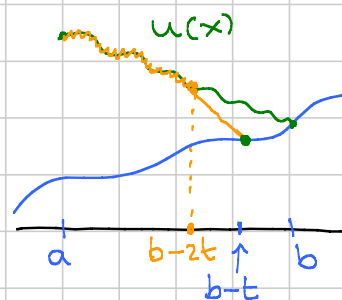
ψ is defined for $t \geq 0$ and has a minimum for $t=0$.
This is enough to conclude that $L(\dots) \geq 0$

2nd step Want to find a new function $u(t, x)$ such that $u(t, b-t) = \varphi(b-t)$

$$u(t, x) := \begin{cases} u(x) & x \in [a, b-2t] \\ \ell(x) & \text{affine interp } x \in [b-2t, b-t] \end{cases}$$

$$\psi(t) := \int_a^{b-2t} L(x, u, \dot{u}) + \int_{b-2t}^{b-t} L(x, \ell(x), \dot{\ell}(x)) dx$$

Compute $\psi'(0) \geq 0$ and obtain... $L(\dots) \leq 0$



In order to compute the derivative of

$$\int_{b-2t}^{b-t} L(x, \ell(x), \dot{\ell}(x)) dx$$

DO DEPEND ON t

in $t=0$, it is simpler to use the definition

||

$$t L(x(t), \ell(x(t)), \dot{\ell}(x(t)))$$

$\downarrow$ $\downarrow$ $\downarrow$
 b $u(b)$ $\dot{u}(b)$

Divide by t and take the limit as $t \rightarrow 0^+$

we need that

$$\dot{u}(b) = \dot{\varphi}(b)$$

— o — o —

Remark The modified function is just piecewise smooth. Let us assume that $u(x)$ is a minimum point in the PWC^1 class.

— o — o —

Detail :

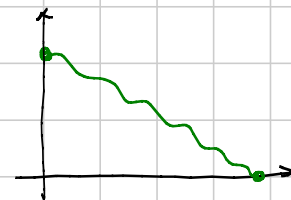
$$\begin{aligned} \dot{\ell}(x) &= \frac{\varphi(b-t) - u(b-2t)}{t} \\ &= \frac{\varphi(b-t) - \varphi(b) + u(b) - u(b-2t)}{t} \end{aligned}$$

$\downarrow$
 $-\dot{\varphi}(b) + 2\dot{u}(b) = \dot{u}(b)$

Example $F(u, b) = \int_a^b \dot{u}^2 dx$ $u(0) = \tau$, $\varphi(x) \equiv 0$

Transversality: $L_p(\dot{u} - \dot{\varphi}) = L$

$$2\dot{u}(b)(\dot{u}(b) - 0) = \dot{u}(b)^2 \Rightarrow \dot{u}(b) = 0$$



Euler equation: $(L_p)' = L_s$ $(2\dot{u})' = 0$ $\dot{u} = \text{const}$

$$\Rightarrow \dot{u} \equiv 0 \Rightarrow u \text{ is constant} \Rightarrow \text{impossible} \Rightarrow \text{NO min in } C^1$$

Actually, the infimum is 0 and a min. seq. is

$$F(u_n(x), n) = \int_0^n \left(\frac{x}{n}\right)^2 dx \rightarrow 0$$



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Calculus of Variations — LECTURE 08

Note Title

07/10/2016

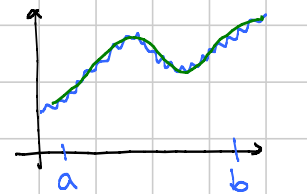
Example 1 (Image processing) Let $f: [a, b] \rightarrow \mathbb{R}$ be given (continuous function)

$$F(u) = \int_a^b \dot{u}^2 + c(u-f)^2 dx$$

↑
positive parameter

Problem: $\min F(u)$ among all $u \in C^1([a, b])$

$f(x) \rightsquigarrow$ NOISY SIGNAL RECEIVED
argmin $\rightsquigarrow$ regularized version of f .



$$L(x, s, p) = p^2 + (s - f(x))^2$$

$$(2\dot{u})' = 2(u - f) \quad \begin{cases} \ddot{u} = u - f \\ \dot{u}(a) = 0 \\ \dot{u}(b) = 0 \end{cases} \leftarrow \text{linear ODE}$$

For ODEs theory $\rightsquigarrow$ existence and uniqueness of solution of class C^2

Since L is convex in (s, p) for every $x \in [a, b]$, the solution is the unique min. point of $F(u)$.

Qualitative properties of the min u :

$$\textcircled{1} \int_a^b u = \int_a^b f \quad \text{Integrate ELE: } \int_a^b \ddot{u} = \int_a^b u - \int_a^b f$$

" $\dot{u}(b) - \dot{u}(a) = 0$

$$\textcircled{2} \max_{x \in [a, b]} u(x) \leq \max_{x \in [a, b]} f(x)$$

$$\min u(x) \geq \min f(x)$$

Proof 1 Look at ELE $\ddot{u} = u - f$

Assume that the maximum point of u is some $x_0 \in (a, b)$

Then

$$0 \geq \ddot{u}(x_0) = u(x_0) - f(x_0)$$

$$\Rightarrow u(x_0) \leq f(x_0) \leq \max_{x \in [a, b]} f(x). \text{ Same for the min.}$$

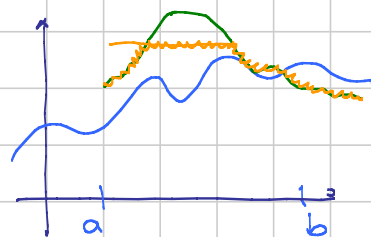
Proof 2 Purely variational proof: TRUNCATION ARGUMENT.

Assume $\max u > \max f$.

I can modify u by truncating with a constant in between.

In this way

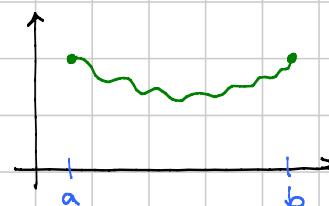
- we save in the term $\int \dot{u}^2$
- we save in the term $\int (u - f)^2$



Example 2 Equilibrium of a string

$$\min \left\{ \int_a^b \dot{u}^2 + k u : u(a) = A, u(b) = B \right\}$$

↑
gravity



$$L(x, s, p) = p^2 + k s \quad (2\dot{u})' = k \quad 2\ddot{u} = k + B C$$

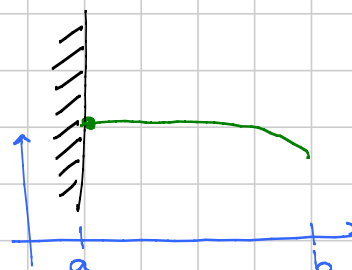
Solution: parabola fitting the boundary conditions.

Example 3 Equilibrium of a BEAM (TRAVE in Italian)

$$\min \left\{ \int_a^b \ddot{u}^2 + k u : u(a) = A, \dot{u}(a) = 0 \right\}$$

$$\psi(t) := \int_a^b (\ddot{u} + t\ddot{v})^2 + k(u + tv)$$

$$\psi'(t) = \int_a^b 2\ddot{u}\ddot{v} + kv \quad \text{1st integral form}$$



$$\begin{aligned}
 &= -\int_a^b 2 \ddot{u} \dot{v} + [\dot{v} \ddot{u}]_a^b + \int_a^b kv \\
 &= \int_a^b 2 u^{(4)} v - [\ddot{u} v]_a^b + [\dot{v} \ddot{u}]_a^b + \int_a^b kv
 \end{aligned}$$

$$\begin{cases} 2 u^{(4)} = k \\ u(a) = A \\ \dot{u}(a) = 0 \\ \ddot{u}(b) = 0 \\ \ddot{u}(b) = 0 \end{cases} \quad \begin{array}{l} \text{ELE} \quad \leadsto u \text{ is pol. of degree 4.} \\ \text{Imposed at the beginning} \\ \text{generated by integr. by parts.} \end{array}$$

Example 6 $F(u) = \int_a^b (\dot{u}^2 + u^2) dx$

with $\{F(u) : u(a) = A, \dot{u}(b) = 1\}$

$$\ddot{u} = u \quad \text{ELE}$$

$$u(a) = A$$

$$\dot{u}(b) = 1 \quad \text{NOT COMPATIBLE}$$

$$\dot{u}(b) = 0$$

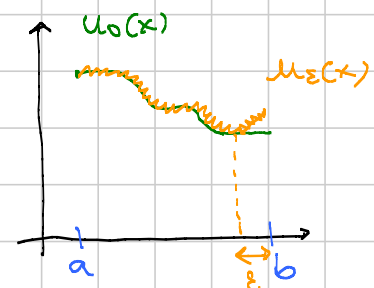
↑ Appears as always

What happens: $\rightarrow$ min does NOT exist

$\rightarrow$ inf = min without condition $\dot{u}(b) = 1$
(which exists because of usual convexity argument)

$\rightarrow$ Take $u_0(x)$ which realizes the minimum

$\rightarrow$ Modify it close to b in order to fulfil $\dot{u}_\varepsilon(b) = 1$.



The same if we add condition $u^{(2016)}(b) = 2017$

The problem has no solution UNLESS u_0 has that property

General fact: functionals with Lagrangians depending on derivatives up to order k ignore BC involving derivatives of order $\geq k$

Example 5 $\int_a^b \dot{u}^4 + u^2$ minimize with DBC

$$(4\dot{u}^3)' = 2u \quad 12\dot{u}^2 \ddot{u} = 2u \quad \leadsto \text{difficult?}$$

ERDMANN form $\dot{u} L_p(u, \dot{u}) = c + L(u, \dot{u})$

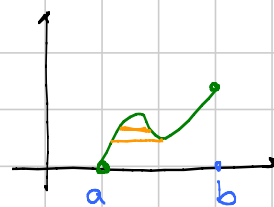
$$4\dot{u}^4 = c + \dot{u}^4 + u^2$$

$$3\dot{u}^4 = c + u^2 \quad \dot{u} = \sqrt[4]{c + \frac{1}{3}u^2}$$

Let us assume that $B > A = 0$

It is reasonable that $u \geq 0$ and $\dot{u} \geq 0$, so

" " " "



$$\dot{u} = \sqrt[4]{c + \frac{1}{3}u^2} \quad \int_a^b \frac{\dot{u}(x)}{\sqrt[4]{c + \frac{1}{3}u(x)^2}} = \int_a^b 1 = b - a$$

$$\begin{aligned} B = u(b) \quad \parallel \quad y = u(x) \quad dy = \dot{u}(x) dx \\ 0 = u(a) \quad \int \frac{dy}{\sqrt[4]{c + \frac{1}{3}y^2}} = b - a \end{aligned}$$

Easy calculus: there is at least (actually exactly) one value of c s.t. $\int = b - a$

The corresponding solution u is the solution to the min. pbm. (Convexity).

Remark There was NO NEED to show $u \geq 0$, $\bar{u} \geq 0$ BEFORE.

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Summary of the INDIRECT METHOD

- ① Derive necessary conditions (ELE and BC)
- ② Hope to be able to solve ELE + BC (or to prove existence of a solution)
- ③ Hope to be able to conclude through
 - convexity
 - trivial lemma.

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Calculus of Variations

LECTURE 09

Note Title

12/10/2016

DIRECT METHOD

Goal: proving that min exists. Then possibly find it.

Given: $f: X \rightarrow \mathbb{R}$ where X is a set.

We need some structure on X :

- set
- set with a NOTION OF CONVERGENCE
- topology
- metric space
- Banach space
- Hilbert space
- $\mathbb{R}^n$

vector structure

Def. A notion of conv. on a set X is "the list of all converging sequences with corresponding limits". More precisely

$$\text{Seq}(X) = \{\text{sequences on } X\} = \{f: \mathbb{N} \rightarrow X\}$$

A notion of convergence in X is a subset of

$$\text{Seq}(X) \times X$$

$$x_m \rightarrow x_\infty$$

We do not ask any special requirement (a sequence can have multiple limits, a subsequence may have a different limit, a constant sequence may have no limit).

Def. A subset $K \subseteq X$ is **COMPACT** wrt to a notion of convergence if every sequence $\{x_n\} \subseteq K$ has a converging subsequence, namely $\exists n_k \rightarrow +\infty$ and $x_\infty \in K$ s.t.

$$x_{n_k} \rightarrow x_\infty$$

Def. A function $f: X \rightarrow \mathbb{R}$ is **lower semi-continuous (LSC)** wrt the notion of convergence if for every $x_n \rightarrow x_\infty$ it turns out that

$$\liminf_{n \rightarrow +\infty} f(x_n) \geq f(x_\infty)$$

The function f is upper semi-continuous (USC) if

$$\limsup_{n \rightarrow +\infty} f(x_n) \leq f(x_\infty)$$

Remark

- If a notion of conv. contains many conv. seqs. then compactness is easy and LSC is difficult
- If it contains few conv. seqs., then the opposite.

THEOREM (Weierstrass) Let $f: X \rightarrow \mathbb{R}$. Let us assume that on X there is a notion of conv. s.t.

- (i) X is compact,
- (ii) f is lower semi-continuous.

Then there exists

$$\min \{f(x) : x \in X\}$$

Proof. For sure there exists $\inf \{f(x) : x \in X\} = I$.

By characterization of inf., there exist a sequence $\{x_n\} \subseteq X$ such that

$$f(x_n) \rightarrow I \in \mathbb{R} \cup \{-\infty\}$$

Since X is compact, there exists a converging subseq.

$$x_{n_k} \rightarrow x_\infty$$

Due to LSC

subseq. of real numbers

$$I = \lim_{n \rightarrow \infty} f(x_n) \stackrel{\downarrow}{=} \lim_{k \rightarrow \infty} f(x_{n_k})$$

$$= \liminf_{k \rightarrow \infty} f(x_{n_k})$$

$$(LSC) \rightarrow \geq f(x_\infty)$$

$$\begin{array}{l} I \text{ is the} \\ \text{infimum} \end{array} \rightarrow \geq I$$

Then we have equality, and therefore $f(x_\infty) = I$. Just in the end we know that $I \in \mathbb{R}$. $\square$

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Def. A function $f: X \rightarrow \mathbb{R}$ is COERCIVE if there exists a compact set $K \subseteq X$ such that

$$\inf \{f(x) : x \in X\} = \inf \{f(x) : x \in K\}$$

Corollary Assume that $f: X \rightarrow \mathbb{R}$ is LSC and coercive. Then the minimum exists.

Proof. $\inf_{x \in X} f(x) = \inf_{x \in K} f(x) = \min_{x \in K} f(x).$

— 0 — 0 —

Corollary Let $f: X \rightarrow \mathbb{R}$. Let us assume that there exists $M \in \mathbb{R}$ such that the SUBLEVEL $\{x \in X : f(x) \leq M\}$ is compact and nonempty. Assume also f is LSC. Then the min exists.

Proof. Previous corollary with $K = \text{sublevel}$. $\square$

HILBERT SPACES

A Hilbert space H is

- a vector space
- has an inner product (symmetric and positive, namely $\langle v, v \rangle > 0$ for every $v \neq 0$)
- it is complete wrt the distance induced by the inner product.

inner product $\rightsquigarrow$ norm $\rightsquigarrow$ distance $\rightsquigarrow$ notion of convergence

$$\|v\|^2 = \langle v, v \rangle \quad \text{dist}(v_1, v_2) := \|v_1 - v_2\|$$

$$v_n \rightarrow v_\infty \Leftrightarrow \text{dist}(v_n, v_\infty) \rightarrow 0.$$

ORTHONORMAL SYSTEM

A subset $\{e_n\} \subseteq H$ is an orth. system (AKA HILBERT BASIS) if

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

and for every $v \in H$ there exists real numbers c_n s.t.

$$v = \sum_{n=0}^{\infty} c_n e_n$$

$\uparrow$
numbers, namely the components
of v wrt $\{e_n\}$

Series in a Hilbert space $\sum_{n=0}^{\infty} v_n = S_\infty$ means that

$$S_n \rightarrow S_\infty \quad \text{where} \quad S_n = v_0 + v_1 + \dots + v_n$$

$\uparrow$ limit in H $\uparrow$ sum in H

Prop. Let $\{e_n\}$ be an orth. system, and let $\{c_n\}$ be a sequence of real numbers.
Then

$$\underbrace{\sum_{n=0}^{\infty} c_n e_n}_{\text{series of vectors}} \text{ converges} \iff \underbrace{\sum_{n=0}^{\infty} c_n^2}_{\text{series of numbers}} \text{ converges}$$

Proof. Let $S_m := c_0 e_0 + \dots + c_m e_m$ $\hat{S}_m := c_0^2 + \dots + c_m^2$
For every $m > n$:

$$\begin{aligned} \|S_m - S_n\|^2 &= \|c_{n+1} e_{n+1} + \dots + c_m e_m\|^2 \\ &= \|c_{n+1}\|^2 + \dots + \|c_m\|^2 \\ &= \hat{S}_m - \hat{S}_n \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n=0}^{\infty} c_n^2 \text{ converges} &\iff \{\hat{S}_n\} \text{ is a Cauchy sequence of numbers} \\ &\iff \{S_n\} \text{ is a Cauchy seq. of vectors} \\ &\iff \underbrace{\sum_{n=0}^{\infty} c_n e_n}_{\text{--- } 0 \text{ --- } 0 \text{ ---}} \text{ converges.} \end{aligned}$$

Def. A Hilbert space H is SEPARABLE if there exists a countable dense subset.

Theorem. Let H be a separable Hilbert space.

Then H admits a (finite or countable) orthonormal system.

Importance of orthonormal systems: the same of the algebraic basis in finite dimension.

In finite dimension, we can identify any vector $v \in V$ with the set of its components.

Every finite dim. vector space V is $\mathbb{R}^n$

In every separable Hilbert space, we can identify each v with the countable set of its components.

In some sense, the space " $\mathbb{R}^\infty$ ".

In particular

$$\|v\|^2 = \sum_{n=0}^{\infty} c_n^2 \quad \text{where} \quad c_n = \langle v, e_n \rangle$$

In other words

$$v = \sum_{n=0}^{\infty} \langle v, e_n \rangle e_n.$$

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Calcolo delle Variazioni

LECTURE 10

Note Title

12/10/2016

STRONG vs WEAK CONVERGENCE

Let H be a separable Hilbert space.

Def. We say that $u_m \rightarrow u_\infty$ **STRONGLY** if

$$\|u_m - u_\infty\| \rightarrow 0 \quad \text{as } m \rightarrow +\infty$$

Good news: the norm, namely the function

$$H \ni u \rightarrow \|u\| \in \mathbb{R}$$

is continuous wrt strong convergence, namely

$$u_m \rightarrow u_\infty \quad \Rightarrow \quad \|u_m\| \rightarrow \|u_\infty\|$$

Proof. $|\|u_m\| - \|u_\infty\|| \leq \|u_m - u_\infty\|$ as a consequence of triangle inequality. $\square$

Bad news If $\dim(H) = +\infty$, the closed balls are NOT COMPACT.

Proof. Take any Hilbert basis $\{e_i\}$. This is a sequence with the property that

$$\text{dist}(e_i, e_j) = \sqrt{2} \quad \text{for every } i \neq j.$$

This forbids the existence of converging subseq. (no subseq. can be a Cauchy sequence). $\square$

Def. We say that u_m WEAKLY converges to u_∞ , and we write

$$u_m \longrightarrow u_\infty$$

$$\text{if } \lim_{m \rightarrow +\infty} \langle u_m, v \rangle = \langle u_\infty, v \rangle \quad \forall v \in H$$

Prop
$$\begin{array}{l} u_m \longrightarrow u_\infty \\ w_m \longrightarrow w_\infty \end{array} \implies \begin{array}{l} u_m + w_m \longrightarrow u_\infty + w_\infty \\ \lambda u_m \longrightarrow \lambda u_\infty \end{array} \quad \forall \lambda \in \mathbb{R}$$

Proof: exercise

Prop. If $u_m \longrightarrow u_\infty$, then $u_m \longrightarrow u_\infty$.

Proof Assume that $u_m \longrightarrow u_\infty$. Then

$$\langle u_m, v \rangle = \langle u_\infty, v \rangle + \underbrace{\langle u_m - u_\infty, v \rangle}_{\downarrow 0}$$

Indeed

$$|\langle u_m - u_\infty, v \rangle| \leq \underbrace{\|u_m - u_\infty\|}_{\downarrow 0} \cdot \|v\|$$

↑ c.s.

If $\dim(H) = +\infty$, then weak convergence does not imply strong convergence

Proof. Let $\{e_n\}$ be any orthonormal basis. I claim that

$$e_m \longrightarrow 0$$

Indeed for every $v \in H$ it turns out that

$$\langle e_m, v \rangle = c_m \quad \text{and} \quad c_m \rightarrow 0 \quad \text{because} \quad \sum_{n=1}^{\infty} c_n^2 < +\infty$$

↑
component of v

Let us apply the definition of weak convergence with $v =$ any element of an orth. system

$$v_m \longrightarrow v_\infty \Rightarrow \forall k \in \mathbb{N} \quad \langle v_m, e_k \rangle \longrightarrow \langle v_\infty, e_k \rangle$$

In other words, weak convergence implies convergence of components.

Actually the converse is true, namely weak convergence is somewhat equivalent to convergence of components.

$$\begin{aligned} \text{In } \mathbb{R}^n : v_m \longrightarrow v_\infty &\Leftrightarrow \text{dist}(v_\infty, v_m) \rightarrow 0 \\ &\Leftrightarrow \text{the components of } v_m \text{ converge to} \\ &\quad \text{the components of } v_\infty \end{aligned}$$

In infinite dimension the two notions are different

$$\begin{array}{ll} \text{dist}(v_\infty, v_m) \rightarrow 0 & \text{IS STRONG CONVERGENCE} \\ \text{conv. of components} & \text{IS WEAK CONVERGENCE.} \\ \text{---} \circ \text{---} \circ \text{---} & \end{array}$$

Very good news : in a separable Hilbert space closed balls are weakly compact

Not so bad news : the norm is no more continuous, but it is at least LSC w.r.t to weak conv.

COMPACTNESS

Let H be a separable Hilbert space

Let $\{v_m\} \subseteq H$ be a bounded sequence, namely

$$\|v_m\| \leq M \quad \forall m \in \mathbb{N}$$

Then there exists $m_k \rightarrow \infty$ and $v_\infty \in H$ such that

$$v_{m_k} \longrightarrow v_\infty$$

Proof. Let us fix an orthonormal system $\{e_k\}$.

STEP 1 Let us consider e_1 . We know that

$\langle v_n, e_1 \rangle$ is a bounded sequence

$$|\langle v_n, e_1 \rangle| \stackrel{\text{C.S.}}{\leq} \|v_n\| \cdot \|e_1\| \stackrel{1}{\leq} M$$

Up to subsequences

$$\langle v_{n_k}, e_1 \rangle \rightarrow c_1 \in \mathbb{R}$$

Let us consider e_2 . Up to further subsequences $\langle v_{n_k}, e_2 \rangle \rightarrow c_2 \in \mathbb{R}$

With a DIAGONAL PROCEDURE we obtain a subsequence v_{n_k} such that

$$\langle v_{n_k}, e_i \rangle \rightarrow c_i$$

$\forall i \in \mathbb{N}$. From now on we assume that

$$\langle v_n, e_i \rangle \rightarrow c_i$$

STEP 2 We would like to define

$$v_\infty := \sum_{i=0}^{\infty} c_i e_i$$

We need to show that the series is convergent, and this is equivalent to

$$\sum_{i=0}^{\infty} c_i^2 < +\infty \quad (\text{series of numbers})$$

which in turn is equivalent to proving that partial sums are equi-bounded. Let $k \in \mathbb{N}$ be fixed. Then

$$M \geq \|v_n\|^2 = \sum_{i=0}^{\infty} \langle v_n, e_i \rangle^2 \geq \sum_{i=0}^k \langle v_n, e_i \rangle^2$$

↓

$$\sum_{i=0}^k c_i^2$$

STEP 3 I claim that $\langle v_n, v \rangle \rightarrow \langle v_\infty, v \rangle$ for every $v \in H$.
To begin with, we prove the result when v is a finite linear combination of elements of $\{e_n\}$.

Let

$$v = a_1 e_1 + \dots + a_k e_k$$

Then

$$\langle v_n, v \rangle = \langle v_n, a_1 e_1 + \dots + a_k e_k \rangle$$

$$= a_1 \langle v_n, e_1 \rangle + \dots + a_k \langle v_n, e_k \rangle$$

↓

↓

$$a_1 \langle v_\infty, e_1 \rangle + \dots + a_k \langle v_\infty, e_k \rangle$$

$$= \langle v_\infty, v \rangle$$

STEP 4 We prove the result for every v . To this end, we fix $\varepsilon > 0$ and we fix $v_\varepsilon \in H$ such that

$$\|v - v_\varepsilon\| \leq \varepsilon \quad \text{and} \quad v_\varepsilon \text{ has a finite number of components.}$$

Now

$$\langle v_n, v \rangle = \boxed{\langle v_n, v_\varepsilon \rangle} + \boxed{\langle v_n, v - v_\varepsilon \rangle}$$

↓

$\langle v_\infty, v_\varepsilon \rangle$

↓

uniformly small

$$|\langle v_n, v - v_\varepsilon \rangle| \leq \underbrace{\|v_n\|}_{\leq M} \cdot \underbrace{\|v - v_\varepsilon\|}_{\leq \varepsilon} \leq M\varepsilon$$

More rigorously

$$\begin{aligned}
 |\langle v_n, v \rangle - \langle v_\infty, v \rangle| &= |\langle v_n, v_\varepsilon \rangle - \langle v_\infty, v_\varepsilon \rangle + \langle v_n, v - v_\varepsilon \rangle| \\
 &\leq \underbrace{\|\langle v_n, v_\varepsilon \rangle - \langle v_\infty, v_\varepsilon \rangle\|}_{\leq \varepsilon \text{ if } n \text{ is large enough.}} + \underbrace{\|\langle v_n, v - v_\varepsilon \rangle\|}_{\leq M\varepsilon}
 \end{aligned}$$

Prop. If $u_n \rightarrow u_\infty$, then

$$\liminf_{n \rightarrow \infty} \|u_n\| \geq \|u_\infty\|$$

Proof.

$$\|u_n\|^2 = \|u_\infty + (u_n - u_\infty)\|^2$$

$$= \|u_\infty\|^2 + \|u_n - u_\infty\|^2 + 2 \langle u_n - u_\infty, u_\infty \rangle$$

$$\geq \|u_\infty\|^2 + 2 \langle u_n - u_\infty, u_\infty \rangle$$

$$= \|u_\infty\|^2 + 2 \langle u_n, u_\infty \rangle - 2 \langle u_\infty, u_\infty \rangle$$

$$\downarrow$$

$$2 \langle u_n, u_\infty \rangle - 2 \langle u_\infty, u_\infty \rangle$$

We proved that

$$\|u_n\|^2 \geq \|u_\infty\|^2 + 2 \langle u_n, u_\infty \rangle - 2 \langle u_\infty, u_\infty \rangle$$

Taking the liminf on both sides we conclude.

Exercise Assume $\{u_n\}$ and $\{w_n\}$ are two sequences.

What do we need to assume in order to conclude that

$$\langle u_n, w_n \rangle \rightarrow \langle u_\infty, w_\infty \rangle$$

Are two weak conv. enough? $u_n \rightarrow u_\infty, w_n \rightarrow w_\infty$

What about $u_n \rightarrow u_\infty$ and $w_n \rightarrow w_\infty$.

Calculus of Variations — LECTURE 11

Note Title

14/10/2016

General setting: H is a separable Hilbert space

$u_m \rightarrow u_\infty$ strongly if $\|u_m - u_\infty\| \rightarrow 0$

$u_m \rightarrow u_\infty$ weakly if $\langle u_m, w \rangle \rightarrow \langle u_\infty, w \rangle \quad \forall w \in H$

Lemma Let u_m and w_m be two sequences in H .

Let us assume that

(i) $u_m \rightarrow u_\infty \in H$,

(ii) $w_m \rightarrow w_\infty \in H$,

(iii) w_m is bounded, namely there exists $M \in \mathbb{R}$ s.t.
 $\|w_m\| \leq M \quad \forall m \in \mathbb{N}$

Then

$$\langle u_m, w_m \rangle \rightarrow \langle u_\infty, w_\infty \rangle.$$

Remark If $u_m \rightarrow u_\infty$ and $w_m \rightarrow w_\infty$, then it may happen that

$$\langle u_m, w_m \rangle \not\rightarrow \langle u_\infty, w_\infty \rangle$$

Example: $u_m = w_m = e_n \leftarrow$ Hilbert basis

$$1 = \langle u_m, w_m \rangle, \text{ but } u_m \rightarrow 0, w_m \rightarrow 0$$

Dim. $|\langle u_m, w_m \rangle - \langle u_\infty, w_\infty \rangle| \leq |\langle u_m, w_m \rangle - \langle u_\infty, w_m \rangle| + |\langle u_\infty, w_m \rangle - \langle u_\infty, w_\infty \rangle|$

$$|\langle u_m - u_\infty, w_m \rangle| \leq \|u_m - u_\infty\| \cdot \underbrace{\|w_m\|}_{\leq M} \leq M \|u_m - u_\infty\| \rightarrow 0$$

$\langle u_\infty, w_m \rangle \rightarrow \langle u_\infty, w_\infty \rangle$ because of the definition of weak convergence. $\square$

Question : if I want to prove that $U_n \rightarrow U_\infty$, can I prove that

$$\langle U_n, w \rangle \rightarrow \langle U_\infty, w \rangle \quad (*)$$

only for some special choices of w ?

Answer ^{If $\{U_n\}$ is bounded, then} It is enough to check $(*)$ for every $w \in W \subseteq H$ with the property that

$$\text{Span}(W) = H \quad \text{in the sense of the distance}$$

Proof. Step 1 If $(*)$ is true for every $w \in W$, then it is true for every $w \in \text{Span}(W)$.

$$\begin{aligned} \langle U_n, c_1 w_1 + \dots + c_k w_k \rangle &= c_1 \langle U_n, w_1 \rangle + \dots + c_k \langle U_n, w_k \rangle \\ &\downarrow \qquad \qquad \qquad \downarrow \\ &= c_1 \langle U_\infty, w_1 \rangle + \dots + c_k \langle U_\infty, w_k \rangle \\ &= \langle U_\infty, c_1 w_1 + \dots + c_k w_k \rangle \end{aligned}$$

Step 2 If $(*)$ is true for every $w \in W$, then it is true for every $w \in \overline{W}$.

Let us take any $w \in \overline{W}$. Let us fix $\varepsilon > 0$. Then there exists $w_\varepsilon \in W$ such that

$$\|w - w_\varepsilon\| \leq \varepsilon.$$

Now

$$\begin{aligned} \underbrace{|\langle U_n, w \rangle - \langle U_\infty, w \rangle|}_{\text{LHS}} &\leq |\langle U_n, w \rangle - \langle U_n, w_\varepsilon \rangle| \quad (1) \\ &\quad + |\langle U_n, w_\varepsilon \rangle - \langle U_\infty, w_\varepsilon \rangle| \quad (2) \\ &\quad + |\langle U_\infty, w_\varepsilon \rangle - \langle U_\infty, w \rangle| \quad (3) \end{aligned}$$

$$(1) = |\langle U_n, w - w_\varepsilon \rangle| \leq \|U_n\| \cdot \|w - w_\varepsilon\| \leq M \cdot \varepsilon$$

$$(2) = \dots \text{ this tends to 0 because } w_\varepsilon \in W$$

$$\textcircled{3} = |\langle v_\infty, w - w_\varepsilon \rangle| \leq \|v_\infty\| \cdot \|w - w_\varepsilon\| \leq \|v_\infty\| \cdot \varepsilon$$

In conclusion

$$\text{LHS} \leq M\varepsilon + \|v_\infty\| \cdot \varepsilon + |\langle v_m, w_\varepsilon \rangle - \langle v_\infty, w_\varepsilon \rangle|$$

and therefore

$$0 \leq \limsup_{n \rightarrow \infty} \text{LHS} \leq M\varepsilon + \|v_\infty\| \cdot \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, then $\limsup \text{LHS} = 0$.

Corollary. If $\{v_m\}$ is a bounded sequence, then $v_m \rightharpoonup v_\infty$ if and only if

$$\langle v_m, e_k \rangle \rightarrow \langle v_\infty, e_k \rangle \quad \forall k \in \mathbb{N}.$$

(components converge to components)

Remark Boundedness is required (if not the result is false)

Example:

$$v_m = m e_m$$

It is obvious that for every FIXED $k \in \mathbb{N}$ it turns out

$$\langle v_m, e_k \rangle = 0 \quad \text{for every } m > k$$

and hence the weak limit is 0, if it exists.

On the other hand, $v_m \not\rightarrow 0$. Why?

We need to find $w \in H$ such that $\langle v_m, w \rangle \rightarrow 0$

Let us take $w = \sum_{n=1}^{\infty} \frac{1}{n} e_n$ $\sum \frac{1}{n^2} < +\infty \Rightarrow w$ is well defined

$$\langle v_m, w \rangle = 1 \quad \text{for every } m \in \mathbb{N}.$$

Theorem (true but not needed in the sequel).

Let us assume that $u_m \rightarrow u_\infty$.

Then $\{u_m\}$ is bounded.

Sketch of proof. We know that $\langle u_m, w \rangle \rightarrow \langle u_\infty, w \rangle$ for every $w \in H$, and therefore

$|\langle u_m, w \rangle|$ is bounded $\forall w \in H$.

$$H_k := \{w \in H : |\langle u_m, w \rangle| \leq k\}$$

Step 1 H_k is closed and $\bigcup_{k \in \mathbb{N}} H_k = H$ (very easy)

Step 2 Baire's Lemma $\Rightarrow \exists k_0$ such that H_{k_0} has nonempty interior, namely it contains a ball.

Step 3 If H_{k_0} has nonempty interior, then it contains a ball with center in the origin

Step 4 Up to changing k_0 , one of the H_k contains the unit ball B with center in the origin, namely

$$\exists k \in \mathbb{N} \text{ s.t. } |\langle u_m, w \rangle| \leq k \quad \forall w \in B$$

Step 5 Let us choose $w = \frac{u_m}{\|u_m\|}$ and obtain that

$$\|u_m\| \leq k \quad \forall m \in \mathbb{N}$$

as required. $\square$

— o — o —

Exercise Let H be a Hilbert space of infinite dimension.
Then H does not admit a countable algebraic basis.

Proof. Let $\{v_n\}$ be a countable algebraic basis. Then

$$H = \bigcup_{k \geq 1} H_k \quad \text{where} \quad H_k = \text{Span} \{v_1, \dots, v_k\}$$

↑
closed subset with empty interior

This is impossible because of Baire's lemma.

Corollary. The space $\mathbb{R}[x]$ cannot become a Hilbert space,
because it admits a countable algebraic basis.

— o — o —

Hopefully trivial comment Let $\{e_n\}$ be a Hilbert basis.
Let us assume that $v_n \rightarrow v_\infty$.

$$\begin{aligned} \|v_n\|^2 &= \langle v_n, e_1 \rangle^2 + \langle v_n, e_2 \rangle^2 + \dots + \langle v_n, e_k \rangle^2 + \dots \\ &\quad \downarrow \qquad \qquad \qquad \downarrow \\ &\quad \langle v_\infty, e_1 \rangle^2 + \langle v_\infty, e_2 \rangle^2 + \dots + \langle v_\infty, e_k \rangle^2 + \dots \\ &= \|v_\infty\|^2 \end{aligned}$$

WRONG because series and term-wise limits DO NOT COMMUTE

— o — o —

Conclusion:

- 1 - There are more weakly conv. seq. than strongly conv. seq.
 - 2 - More conv. seq. $\Rightarrow$ easier to be compact (and indeed balls are weakly compact but not strongly compact.)
 - 3 - More conv. seq. $\Rightarrow$ more diff. to have continuity
(the norm is strongly continuous, but only weakly LSC)
- o — o —

Calculus of Variations — LECTURE 12

Note Title

14/10/2016

Some basic facts about Lebesgue measure and Lebesgue spaces.

Passing to the limit in integrals

① BEPPO LEVI (monotone convergence theorem)

$$f_n(x) \geq 0 \text{ with } f_{n+1}(x) \geq f_n(x) \quad \forall x \in (a,b) \quad \forall n \in \mathbb{N}$$

Then

$$\int_a^b \underbrace{\sup_{n \in \mathbb{N}} f_n(x)}_{\downarrow \lim_{n \rightarrow \infty} f_n(x)} dx = \sup \int_a^b f_n(x) dx$$

② FATOU'S LEMMA

$$f_n(x) \geq 0 \quad \forall n \in \mathbb{N} \quad \forall x \in (a,b)$$

Then

$$\int_a^b \liminf_{n \rightarrow \infty} f_n(x) dx \leq \liminf_{n \rightarrow \infty} \int_a^b f_n(x) dx$$

③ DOMINATED CONVERGENCE THEOREM (Lebesgue's theorem)

Let us assume that

DOMINATION

$$|f_n(x)| \leq g(x) \quad \forall x \in (a,b) \quad \forall n \in \mathbb{N}$$

where $\int_a^b g(x) dx < +\infty$

Let us assume that $f_n(x) \rightarrow f_\infty(x) \quad \forall x \in (a,b)$
(pointwise convergence)

Then

$$\int_a^b f_\infty(x) dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x) dx$$

LEBESGUE SPACES

Let $(a,b) \subseteq \mathbb{R}$ be an interval, and let $p \geq 1$.

The space $L^p(a,b)$ is the set of all functions $f: (a,b) \rightarrow \mathbb{R}$ such that

$$\int_a^b |f(x)|^p dx < +\infty$$

We set

$$\|f\|_p := \left[\int_a^b |f(x)|^p dx \right]^{1/p}$$

Main properties:

- L^p is a vector space
- $\|f\|_p$ is a norm
- L^p is complete wrt the norm, and therefore it is a Banach space (this point requires Lebesgue's measure)

Special case $p=2$: $L^2(a,b)$ is a Hilbert space with inner product

$$\langle f, g \rangle := \int_a^b f(x) \cdot g(x) dx$$

HÖLDER'S INEQUALITY

$f \in L^p(a,b)$, $g \in L^q(a,b)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$fg \in L^1$ and $\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q$, namely

$$\left| \int_a^b f(x) g(x) dx \right| \leq \left[\int_a^b |f(x)|^p dx \right]^{1/p} \cdot \left[\int_a^b |g(x)|^q dx \right]^{1/q}$$

Special case $p=q=2$: Hölder's ineq. is just Cauchy-Schwarz in L^2 .

More general statement : Let us assume that

$$\frac{1}{p_1} + \dots + \frac{1}{p_k} = \frac{1}{r}$$

and let $f_i \in L^{p_i}$ for every $i=1, \dots, k$.

Then $f_1 \cdot f_2 \cdot \dots \cdot f_k \in L^r$ and

$$\|f_1 \cdot \dots \cdot f_k\|_r \leq \|f_1\|_{p_1} \cdot \dots \cdot \|f_k\|_{p_k}.$$

Approximation results

For every $f \in L^p$, there exists a sequence $\{f_m\} \subseteq C^\infty(a,b)$ such that

$$f_m \rightarrow f \quad \text{in } L^p$$

In the same way we can approximate with piece-wise constant functions or with C^∞ functions satisfying a finite number of conditions in points of (a,b)

Fundamental Lemma in the Calc Var (for L^1 functions)

Let $f \in L^1(a,b)$ be such that

$$\int_a^b f(x) v(x) dx = 0 \quad \forall v \in C_c^\infty(a,b)$$

Then $f(x) = 0$ for every $x \in (a,b)$ (up to a set of meas = 0)

Proof. Let us take a sequence $\{v_m\} \in C_c^\infty$ such that

$$v_m \rightarrow f \quad \text{in } L^1(a,b)$$

Then
$$\int_a^b f(x) v_m(x) dx = 0 \quad \forall m \in \mathbb{N}$$

$$\downarrow$$

$$\int_a^b f(x) f(x) dx = 0 \Rightarrow f(x) \equiv 0 \text{ up to a set of measure 0.}$$

Remark 1 In the approx result we can assume also that the convergence $f_m \rightarrow f$ is in L^p , but also pointwise and dominated. Therefore the convergence of the integrals in the previous proof follows from Lebesgue's theorem.

Remark 2 Since (a,b) is a bounded interval, then

$$L^p \supseteq L^q \quad \text{for every } p < q$$

So the FLCV is true in every L^p space.

Remark 3 DBR lemma can be proved in the same way.

Useful fact L^p is a separable space for every $p \geq 1$

Possible countable dense subsets

→ Polynomials with rational coefficients (requires Stone-Weierstrass)

→ linear combinations with rational coeff. of characteristic functions of intervals with rational endpoints.

Consequence: L^2 admits a (actually many) orthonormal systems.

Example: a orthonormal system in $L^2((0, \pi))$ is $\{\sin(nx)\}_{n \in \mathbb{N}}$

up to normalization constants.

(The only nontrivial part is that their span is dense).

The same is true with $\{\cos(nx)\}_{n \in \mathbb{N}}$

Weak convergence in $L^2(a, b)$ is $\forall w \in L^2$

$$f_n \rightarrow f \iff \int_a^b f_n(x) w(x) dx \rightarrow \int_a^b f(x) w(x) dx$$

Example 1 $f_n(x) = \sin(nx)$ weakly converges to 0 in $L^2((0, \pi))$

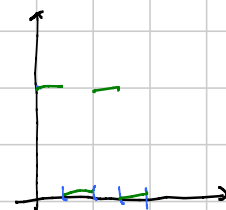
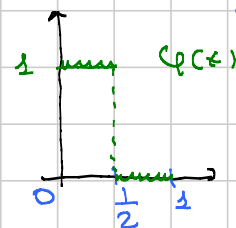
Proof. It is enough to check the relation for every w which is a characteristic function of a subinterval.

$$\int_0^\pi \sin(nx) \cdot \mathbb{1}_{(a,b)}(x) dx = \int_a^b \sin(nx) dx = \frac{1}{n} \cdot \text{something} \xrightarrow{\downarrow} 0$$

Example 2 $(a, b) = (0, 1)$

$f_n(x) = \varphi(nx)$ = "greek functions"

$f_n(x) \rightarrow f_\infty(x) \equiv \frac{1}{2}$ (proof as before)



Calculus of Variations - LECTURE 13

Note Title

19/10/2016

SOBOLEV SPACES (In dimension 1)

Def. (w) let $f \in L^1(\mathbb{R})$ and let $g \in L^1(\mathbb{R})$.

We say that g is the weak derivative of f if

$$\int_{\mathbb{R}} f(x) \varphi'(x) dx = - \int_{\mathbb{R}} g(x) \varphi(x) dx \quad \forall \varphi \in C_c^\infty(\mathbb{R})$$

We say that $f \in W^{1,p}(\mathbb{R})$ if $f \in L^p(\mathbb{R})$ and f has a weak derivative $g \in L^p(\mathbb{R})$.

Remark In the same way we can define k -th order derivatives and $W^{k,p}(\mathbb{R})$

$$f \in W^{k+1,p}(\mathbb{R}) \Leftrightarrow f \in W^{k,p}(\mathbb{R}) \text{ and the weak derivative } g \in W^{k,p}(\mathbb{R})$$

Remark In the same way we can define $W^{1,p}(a,b)$. In this case

$$\int_a^b f(x) \varphi'(x) dx = - \int_a^b g(x) \varphi(x) dx \quad \forall \varphi \in C_c^\infty(a,b)$$

PROPERTIES - EXAMPLES

① Uniqueness The weak derivative, when it exists, is unique.

Proof. Let g_1 and g_2 be two weak derivatives of f in (a,b) . Then

$$\int_a^b f(x) \varphi'(x) dx = - \int_a^b g_1(x) \varphi(x) dx = - \int_a^b g_2(x) \varphi(x) dx \quad \forall \varphi \in C_c^\infty(a,b)$$

Then $\int_a^b [g_1(x) - g_2(x)] \varphi(x) dx = 0 \quad \forall \varphi \in C_c^\infty(a, b)$

Conclusion by FLCV.

② $W^{1,p}(a, b)$ is a vector space and the weak derivative is a linear function.

Proof. Easy exercise

③ Consistency with classical notions

[3.1] If $f \in C^1(a, b)$, then $g = f'$ (integration by parts)

[3.2] If the weak derivative of f is some $g \in C^0(a, b)$, then actually f is of class C^1 , and coincides with some antiderivative of g .

Proof. Let $G(x)$ be any antiderivative of $g(x)$. Then

$$\int_a^b f(x) \varphi'(x) dx = - \int_a^b g(x) \varphi(x) dx = \int_a^b G(x) \varphi'(x) dx$$

$\uparrow$ defn. of g
 $\uparrow$ int. by parts (classical)

$$\Rightarrow \int_a^b [f(x) - G(x)] \varphi'(x) dx = 0 \quad \forall \varphi \in C_c^\infty(a, b)$$

DBR Lemma $\Rightarrow f(x) - G(x)$ is constant.

④ $f(x) = |x|$ with $(a, b) = (-1, 1)$. Then it has a weak derivative which is

$$g(x) = \begin{cases} -1 & x \in (-1, 0) \\ 1 & x \in (0, 1) \end{cases}$$

Proof.

$$\begin{aligned}
 \int_{-1}^1 f(x) \varphi'(x) dx &= \int_{-1}^0 (-x) \varphi'(x) dx + \int_0^1 x \varphi'(x) dx \\
 &= \underbrace{\int_{-1}^0 1 \cdot \varphi(x) dx}_{\text{blue}} + [-x \cdot \varphi(x)]_{-1}^0 \\
 &\quad + \underbrace{[x \varphi(x)]_0^1 - \int_0^1 1 \cdot \varphi(x) dx}_{\text{blue}} \\
 &= - \int_{-1}^1 g(x) \varphi(x) dx
 \end{aligned}$$

In an analogous way, all functions which are continuous and PWC^1 do admit a weak derivative

Proof. Easy exercise.

⑤ $f(x) = \text{Heaviside function} = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$

This function has no weak derivative $g(x)$ in $(-1, 1)$

Proof. Let g be the weak derivative of f . Then

$$\int_{-1}^1 f(x) \varphi'(x) dx = - \int_{-1}^1 g(x) \varphi(x) dx \quad \forall \varphi \in C_c^\infty((-1, 1))$$

$$\int_0^1 \varphi'(x) dx = -\varphi(0)$$

$$\Rightarrow \int_{-1}^1 g(x) \varphi(x) dx = \varphi(0) \quad \forall \varphi \in \dots$$

$$\int_{-1}^1 g(x) \varphi_n(x) dx = \varphi_n(0) = 1 \quad \forall n \in \mathbb{N}$$

$\downarrow$ $\downarrow$
 0 $\swarrow$ pointwise limit + domination
 + dominated convergence theorem.



Def. (H) We say that $f \in H^{1,p}((a,b))$ and $g \in L^p((a,b))$ is its weak derivative if there exists a sequence $\{f_n\} \subseteq C^\infty((a,b))$ (not asking compact support) such that

$$f_n \rightarrow f \text{ in } L^p((a,b)) \quad f_n' \rightarrow g \text{ in } L^p((a,b))$$

Same defn. in more abstract setting

Let us consider $C^1((a,b))$ or $C^\infty((a,b))$ with the norm

$$\|f\|_p := \|f\|_p + \|f'\|_p$$

The norm induce a structure of metric space. Then $H^{1,p}((a,b))$ is the completion of this metric space.

— o — o —

Properties

① Uniqueness $\leadsto$ not so clear

② Vector space $\leadsto$ easy exercise

③ Consistency with classical case

③.1 Easy if we admit $f_n \in C^1((a,b))$

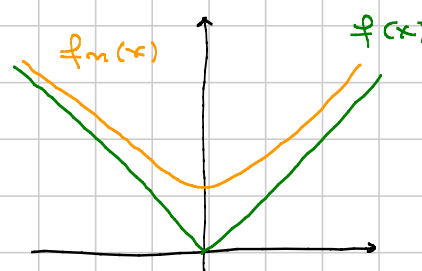
③.2 Less easy

④ $f(x) = |x| \in H^{1,p}((-1,1))$

Can take $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$

Easy to check that $f_n \rightarrow f$ in $L^p((-1,1))$

$f_n' \rightarrow -1/1$ in $L^p((-1,1))$



⑤ Heaviside function $\notin H^{1,p}((-1,1)) \rightsquigarrow$ not so clear.

Theorem $f \in W$ (and g is the weak derivative)

Proof. We know that $f_n \rightarrow f$ and $f_n' \rightarrow g$ in $L^p((a,b))$
For every $n \in \mathbb{N}$:

$$\int_a^b f_n(x) \varphi'(x) dx = - \int_a^b f_n'(x) \varphi(x) dx \quad \forall \varphi \in \dots$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$\int_a^b f(x) \varphi'(x) dx = - \int_a^b g(x) \varphi(x) dx$$

Both integrals converge because of dominated convergence.

Corollary All the missing points in the previous properties have now been proved.

Theorem let us assume that $\{f_n\} \subseteq W^{1,2}((a,b))$ and let $\{g_n\} \subseteq L^2((a,b))$ be the sequence of weak derivatives. Let us assume that

$$\begin{array}{ccc} f_n \longrightarrow f_\infty & \text{WEAKLY in } L^2((a,b)) \\ g_n \longrightarrow g_\infty & \text{" " "} \end{array}$$

Then $f_\infty \in W^{1,2}((a,b))$ and g_∞ is the weak derivative of f_∞ .

Proof.

$$\int_a^b f_n(x) \varphi'(x) dx = - \int_a^b g_n(x) \varphi(x) dx \quad \forall \varphi \in C_c^\infty$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$\int_a^b f_\infty(x) \varphi'(x) dx = - \int_a^b g_\infty(x) \varphi(x) dx$$

Notation: $H^k((a,b)) = H^{k,2}((a,b)) = W^{k,2}((a,b))$

Calculus of Variations — LECTURE 14

Note Title

19/10/2016

Sobolev functions are antiderivatives of their weak derivative

Let $f \in L^p(a,b)$, and let $g \in L^p(a,b)$ be the weak derivative.
Let us set

$$G(x) := \int_a^x g(s) ds \quad \forall x \in (a,b)$$

Proposition $f(x) - G(x) = \text{constant}$ (up to a set of measure = 0)

Proof. $\int_a^b \varphi'(x) G(x) dx = \int_a^b \varphi'(x) \int_a^x g(s) ds dx$

$$= \int_a^b \int_a^x \varphi'(x) g(s) ds dx = \iint_T \varphi'(x) g(s) dx ds$$

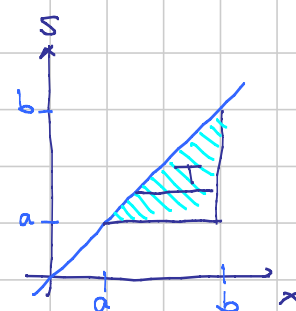
$$= \int_a^b ds \int_s^b dx \varphi'(x) g(s)$$

(Fubini
Tonelli)

$$= \int_a^b g(s) ds \int_s^b \varphi'(x) dx$$

$$= - \int_a^b g(s) \varphi(s) ds = - \int_a^b g(x) \varphi(x) dx$$

defn. of $\int_a^b f(x) \varphi'(x) dx$



Compare LHS and RHS + DBR lemma $\Rightarrow G - f$ is constant.

— 0 — 0 —

Theorem $W^{1,p} \subseteq H^{1,p}$ both in (a,b) and in $\mathbb{R}$

Proof. Assume that $f \in W^{1,p}$ and g is the weak derivative.

We can approximate g with $C^\infty \ni g_m \rightarrow g$ in L^p

We can define f_m as antiderivatives of g_m and we have that

$$\begin{array}{ccc} g_m \rightarrow g & \text{in } L^p & f_m \rightarrow f & \text{in } L^p \\ \text{"} \text{"} & & & \\ f_m' & & & \end{array}$$

— o — o —

HÖLDER CONTINUITY OF SOBOLEV FUNCTIONS

Assume that $f \in W^{1,p}(a,b)$ and g is the weak derivative. Then f is $\frac{1}{p}$ -Hölder continuous, namely there exists M s.t.

$$|f(x_1) - f(x_2)| \leq M |x_1 - x_2|^{1/p} \quad \forall x_1, x_2 \in (a,b)$$

Proof. We know that $f(x)$ is "the" antiderivative of g , and hence

$$|f(x_2) - f(x_1)| = \left| \int_{x_1}^{x_2} g(s) ds \right|$$

$$\begin{array}{c} | \\ a \quad x_1 \quad x_2 \quad b \end{array}$$

$$\leq \int_{x_1}^{x_2} |g(s)| \cdot 1 \, ds$$

$\in L^p \quad \in L^q$

$$\frac{1}{p} + \frac{1}{q} = 1$$

(Hölder's ineq.) $\leq \left[\int_{x_1}^{x_2} |g(s)|^p ds \right]^{1/p} \cdot \left[\int_{x_1}^{x_2} 1^q ds \right]^{1/q}$

$$\leq \left[\int_a^b |g(s)|^p ds \right]^{1/p} |x_2 - x_1|^{1/q}$$

$$= \|g\|_{L^p(a,b)} \cdot |x_2 - x_1|^{1/q}$$

Result

$$|f(x_2) - f(x_1)| \leq \|g\|_{L^p([a,b])} \cdot |x_2 - x_1|^{1/q}$$

Special case If $p=q=2$, we have that functions in $H^1([a,b])$ or $W^{1,2}([a,b])$ is $\frac{1}{2}$ -Hölder continuous.

— 0 — 0 —

ASCOLI - (ARZELÀ) THEOREM

Let $f_n: [a,b] \rightarrow \mathbb{R}$. Let us assume that

(i) $\{f_n\}$ are **EQUIBOUNDED**, namely there exists $M \in \mathbb{R}$ s.t.

$$|f_n(x)| \leq M \quad \forall n \in \mathbb{N}, \forall x \in [a,b]$$

(ii) $\{f_n\}$ are **EQUICONTINUOUS**, namely

$$\forall \varepsilon > 0 \quad \forall x \in [a,b] \quad \exists \delta > 0 \quad \text{s.t.}$$

$$|f_n(x) - f_n(y)| \leq \varepsilon \quad \forall n \in \mathbb{N} \quad \forall y \in [x-\delta, x+\delta] \cap [a,b].$$

Then $\{f_n\}$ admits a subsequence converging uniformly, namely there exists $n_k \rightarrow \infty$ and $f_\infty: [a,b] \rightarrow \mathbb{R}$ such that

$$f_{n_k} \rightarrow f_\infty \quad \text{uniformly.}$$

— 0 — 0 —

More general statement : $f_n: X \rightarrow Y$, with X compact.

$\uparrow$ $\uparrow$
 metric spaces metric spaces

Assumption (ii) is the same, and (i) becomes

(i-bis) $\forall x \in X \quad \exists K_x \subseteq Y$ compact such that

$$f_n(x) \in K_x \quad \forall n \in \mathbb{N} \quad \forall x \in X.$$

Remark Assume that $\{f_n\}$ are equi-Hölder continuous, namely

$$|f_n(x_2) - f_n(x_1)| \leq M \cdot |x_2 - x_1|^\alpha \quad \forall n \in \mathbb{N}, \forall x_1, x_2 \in \dots$$

with the same M and the same $\alpha \in (0, 1]$ for all n 's.
Then equicontinuity follows. $\underline{\quad} \quad \underline{\quad} \quad \underline{\quad}$

Let's go back to defn. (H)

$$f_n \rightarrow f \text{ in } L^p(a, b) \quad f'_n \rightarrow g \text{ in } L^p(a, b)$$

Use only that $\|f_n\|_{L^p(a, b)}$ is bounded. Use previous estimate

$$\begin{aligned} |f_n(x_2) - f_n(x_1)| &\leq \|f_n\|_{L^p(a, b)}^{1/q} |x_2 - x_1|^{1/q} \\ &\leq M |x_2 - x_1|^{1/q} \end{aligned}$$

This proves that the family $\{f_n\}$ is equicontinuous.

We need equiboundedness: $f_n \rightarrow f$ in $L^p(a, b)$, so again

$$\|f_n\|_{L^p(a, b)} \text{ is bounded, therefore } \int_a^b |f_n(x)|^p dx \leq M$$

" mean value theorem

$$(b-a) |f_n(x_m)|^p$$

$$\Rightarrow |f_n(x_m)| \leq M_1 \Rightarrow f_n \text{ is equibounded in some point } x_m.$$

For any $x \in [a, b]$ we have

$$\begin{aligned} |f_n(x)| &\leq |f_n(x_m)| + |f_n(x) - f_n(x_m)| \\ &\leq M_1 + M |x - x_m|^{1/q} \\ &\leq M_1 + M |b - a|^{1/q} \end{aligned}$$

In conclusion $\{f_n\}$ satisfies the assumptions of AA. theorem, and therefore (up to subsequences)

$$f_{n_k} \rightarrow f_\infty \text{ uniformly and the limit is continuous}$$

↓
same limit as in L^p

At this point we know also that the whole sequence $f_n \rightarrow f$ uniformly.

Remark This provides alternative proof that Sobolev functions are Hölder cont.

— o — o —

Assume $f \in W^{1,p}((a,b))$. Conjecture: $\sin f \in W^{1,p}((a,b))$
and $(\sin f)' = \cos f \cdot g$

"Exercise": prove it through definition (w)

Using (H) it's easier $f_n \rightarrow f \quad f_n' \rightarrow g$

$$\sin f_n \rightarrow \sin f \quad (\sin f_n)' = \cos f_n \cdot f_n' \rightarrow \cos f \cdot g$$

— o — o —

Relation weak derivative — difference quotient

$$Q_R(x) = \frac{f(x+R) - f(x)}{R} \quad f: \mathbb{R} \rightarrow \mathbb{R}$$

Useful facts (exercises in the case $p=2$)

$$f \in W^{1,p} \Leftrightarrow Q_R(x) \rightarrow g \text{ in } L^p \Leftrightarrow Q_R(x) \rightarrow g \text{ in } L^p$$

$\Downarrow$
 $\|Q_R\|_{L^p}$ equibounded
indep. of R .

— o — o —

Calculus of Variations – LECTURE 15

Note Title

21/10/2016

DIRECT METHOD

- ROAD MAP
- ① Weak formulation : finding a suitable functional setting (Sobolev spaces)
 - ② Compactness : finding a suitable notion of convergence wrt which sublevels are compact.
 - ③ Lower semi-continuity : of the functional with respect to the previous notion of conv.
 - ④ Regularity : back to classical solutions, usually solutions are more regular than expected in ①.

Example $F(u) = \int_0^1 [\dot{u}^2 + \cos(u^3)] dx$

Pbm: $\min \{ F(u) : u \in C^1([0,1]), u(0)=2, u(1)=7 \}$

- ① Weak formulation Extend the function to $H^1((0,1))$ because in this space

$$\int_0^1 \dot{u}^2 dx$$

is well-defined and finite.

Remark If $u \in H^1((0,1))$, then u is continuous in $[0,1]$ so that pointwise conditions such as DBC make sense for u .

On the contrary, pointwise conds such as $\dot{u}(c) = C$ make no sense in H^1 because $\dot{u}$ is just L^2 .

② Compactness of sublevels. We need a notion of conv. s.t.

$\{u \in H^1 : u \text{ satisfies } \mathcal{B}C \text{ and } F(u) \leq M\}$
is compact.

A reasonable choice is the following: we say that $\{u_n\}$ converges to some u_∞ if

$$u_n \rightarrow u_\infty \text{ unif. in } [0,1] \quad \dot{u}_n \rightarrow \dot{u}_\infty \text{ weakly}$$

Remark If we ask that $\{\dot{u}_n\}$ is bounded in $L^2(0,1)$, then the sequence $\{u_n\}$ is equicontinuous and therefore any convergence on u_n (for example $u_n \rightarrow u_\infty$ in L^2 or even $u_n \rightarrow u_\infty$ weakly) implies unif. conv.

Let us take a sublevel

$$\int_0^1 [\dot{u}^2 + \underbrace{\cos(u^3)}_{\geq -1}] \leq M$$

$$\Rightarrow \int_0^1 \dot{u}^2 dx \leq M - \int_0^1 \cos(u^3) dx \leq M+1$$

If we take any sequence $\{u_n\} \subseteq \text{sublevel}$, then

$$\|\dot{u}_n\|_{L^2}^2 \leq M+1$$

and therefore there exists $\dot{u}_{n_k} \rightarrow v_\infty$ weakly

In order to prove unif. conv. of u_{n_k} (up to further subseq.) we want to apply Ascoli - Arzelà. So we need to check

- (i) equiboundedness
- (ii) equicontinuity

Equicontinuity follows from the bound on derivatives

$$|u_n(x) - u_n(y)| \leq \underbrace{\|u_n'\|_{L^2}}_{\text{bounded}} \cdot |x-y|^{1/2} \quad \forall (x,y) \in [0,1]^2$$

(see lecture 14)

Equiboundedness follow from BC + equicontinuity

$$|u_n(x)| \leq \underbrace{|u_n(0)|}_{\substack{'' \\ \text{(B.C)}}} + \underbrace{|u_n(x) - u_n(0)|}_{\text{bounded because of previous estimate}}$$

A.A. $\Rightarrow u_{n_k}$ has a subseq. (not relabeled) such that

$$u_{n_k} \rightarrow u_\infty \text{ unif.}$$

We need that $v_\infty = u_\infty'$ (weak derivative) (for this it is enough to have weak conv. of functions and derivatives)

③ Lower semicontinuity Need to prove that

$$\left. \begin{array}{l} u_n \rightarrow u_\infty \text{ unif} \\ u_n' \rightarrow u_\infty' \end{array} \right\} \Rightarrow \liminf_{n \rightarrow \infty} F(u_n) \geq F(u_\infty)$$

Let us consider $\int_0^1 \cos(u_n^3) dx$

This converges to $\int_0^1 \cos(u_\infty^3) dx$ because

$$\cos(u_n^3) \rightarrow \cos(u_\infty^3) \text{ uniformly in } [0,1].$$

Let us consider $\int_0^1 u_n'^2 dx$. This is the square of the norm of u_n in $L^2((0,1))$, and the norm is LSC, and hence

$$\liminf_{n \rightarrow \infty} \int_0^1 u_n^2 \geq \int_0^1 u_\infty^2$$

In conclusion

$$\underbrace{\int_0^1 u_n^2}_{\substack{\downarrow \\ \text{liminf} \\ \text{with the} \\ \text{right ineq.}}} + \underbrace{\int_0^1 \cos(u_n^3)}_{\downarrow \text{limit}}$$

Summary After ①, ②, ③ we know that the minimum problem admits at least solution in $H^1([0,1])$.

Remark The limit function u_∞ satisfies the BC because of unif. conv.

④ Regularity Let u_0 be a minimum point in H^1 . Let v be any function in $C_c^\infty([0,1])$. We can compute the first variation in H^1 :

$$\psi(t) := F(u_0 + tv) := \int_0^1 (u_0 + tv)^2 + \cos(u_0 + tv)^3$$

$$\psi'(0) = \int_0^1 2u_0 \dot{v} - \sin(u_0^3) \cdot 3(u_0)^2 v = 0$$

↑
this requires computing the derivative of an integral depending on a parameter.

$$\underbrace{\int_0^1 2u_0 \dot{v}}_f = \int_0^1 \underbrace{\sin(u_0^3) 3u_0^2}_{-g} \cdot v \quad \forall v \in C_c^\infty([0,1])$$

This is equivalent to saying that f admits a weak derivative and this weak derivative is actually g .

Therefore $u_0 \in H^1$, and hence $u_0 \in H^2$ and

$$2 \ddot{u}_0 = -3u_0^3 - 3u_0^2$$

Diff. form of Euler equ.

↑
weak second
order derivative

BOOTSTRAP ARGUMENT

$u_0 \in H^2 \leadsto u_0$ is cont. $\leadsto$ RHS is cont. $\leadsto \ddot{u}_0$ is cont
 $\leadsto u_0 \in C^2 \leadsto \text{RHS} \in C^2 \leadsto \ddot{u}_0 \in C^2$
 $\leadsto u_0 \in C^4 \leadsto \dots$

and so on.

Conclusion: any minimum point of $F(u)$ in H^1 is actually of class C^∞ , and it is a classical solution of the usual Euler equ.

Summary: usually regularity has two steps

- from H^1 to H^2 (here we exploit the defn. of weak derivative)
- in H^2 we can write Euler equ. in diff. form
- from H^2 to C^∞ (easy bootstrap argument)

Calculus of Variations - LECTURE 16

Note Title

21/10/2016

Compactness theorems

Almost any compactness theorem in function spaces has 2 main ingredients:

- something that prevents functions from becoming too big
- " " " too many oscillations.

Examples

$u_n(x) := n$ (no oscillations, but no control on functions)

$u_n(x) := \sin(nx)$ (bounded functions, but no control on oscillations).

A bound on derivatives usually gives a bound on oscillations.

Typical result Let $u_n : [a, b] \rightarrow \mathbb{R}$.

Let us assume that

(i) $\|u_n\|_{L^2([a, b])} \leq M$ ($u_n \in H^1([a, b])$)

(ii) $\forall n \in \mathbb{N} \exists x_n \in [a, b] \text{ s.t. } |u_n(x_n)| \leq M$

Then there exists a subsequence such that

$$u_{n_k} \rightarrow u_\infty \text{ unif.} \quad \dot{u}_{n_k} \rightarrow \dot{u}_\infty$$

Proof. Follows from A.A. where

- equicontinuity follows from (i)
- equiboundedness follows from (i) and (ii)

$$|u_n(x)| \leq \overbrace{|u_n(x_n)|}^{\text{bounded by (ii)}} + \overbrace{|u_n(x) - u_n(x_n)|}^{\text{bounded due to (i)}} \quad \square$$

What gives (ii) in the case of functionals

→ a DBC in an endpoint

→ some integral constraint such as $\int_a^b u_n dx = \text{constant}$

or $\int_a^b |u_n(x)|^p dx = \text{constant}$

Proof. Use mean value theorem for integrals

$$(b-a) u_n(x_n) = \int_a^b u_n(x) dx = \text{constant}$$

Lower semicontinuity

Terms involving u

Assume that $u_n \rightarrow u_\infty$ unif. and $\psi: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Then

$$\lim_{n \rightarrow \infty} \int_a^b \psi(u_n(x)) dx = \int_a^b \psi(u_\infty(x)) dx$$

Proof. Enough to show that $\psi(u_n) \rightarrow \psi(u_\infty)$ unif. in $[a,b]$. This exploits the unif. cont. of ψ on compact sets.

More generally, we can treat in the same ways also

$$\int_a^b \psi(x, u_n(x)) dx \quad \psi: [a,b] \times \mathbb{R} \rightarrow \mathbb{R}.$$

More general case: assume that $\psi: \mathbb{R} \rightarrow \mathbb{R}$ is just LSC and bounded from below. Then

$$\liminf_{n \rightarrow \infty} \int_a^b \psi(u_n(x)) dx \geq \int_a^b \psi(u_\infty(x)) dx \quad [\text{Proof: Fatou's Lemma}]$$

Terms involving u

Before we exploited that $\liminf_{n \rightarrow \infty} \int_a^b u_n^2 \geq \int_a^b u_\infty^2$
provided that $u_n \rightarrow u_\infty$.

Proof (again): assume that $v_n \rightarrow v_\infty$. Then

$$\begin{aligned} \int_a^b v_n^2 &= \int_a^b [v_\infty - (v_\infty - v_n)]^2 \\ &= \int_a^b v_\infty^2 - 2 \underbrace{\int_a^b v_\infty (v_\infty - v_n)}_{\downarrow 0 \text{ as } n \rightarrow \infty \text{ because of weak convergence}} + \underbrace{\int_a^b (v_\infty - v_n)^2}_{\geq 0} \end{aligned}$$

What about the case

$$\int_a^b \psi(v_n)$$

Theorem Assume that $\psi: \mathbb{R} \rightarrow \mathbb{R}$ is convex
Assume that $v_n \rightarrow v_\infty$ in $L^2(a,b)$

Then

$$\liminf_{n \rightarrow \infty} \int_a^b \psi(v_n) \geq \int_a^b \psi(v_\infty)$$

Proof. Let us cheat a little bit...

Due to convexity it turns out that

$$\psi(a+b) \geq \psi(a) + \psi'(a)b \quad \forall a \in \mathbb{R} \quad \forall b \in \mathbb{R}$$

Apply this inequality with $a = v_\infty(x)$, $b = v_n(x) - v_\infty(x)$:

$$\int_a^b \underbrace{\psi(v_n(x))}_{a+b} \geq \int_a^b \underbrace{\psi(v_\infty(x))}_a + \int_a^b \underbrace{\psi'(v_\infty(x))}_a \underbrace{(v_n(x) - v_\infty(x))}_b$$

Taking the limit

$$\lim_{n \rightarrow \infty} LHS = \int_a^b \psi(v_\infty(x)) + \lim_{n \rightarrow \infty} \underbrace{\int_a^b \psi'(v_\infty) \cdot (v_n - v_\infty)}_{\substack{\downarrow 0 \\ \text{because } v_n \rightarrow v_\infty}}$$

Remark ① We assumed that ψ is of class C^1
 ② We assumed that $\psi'(v_\infty) \in L^2((a,b))$.
 A sufficient condition is that ψ' has linear growth, namely $\exists A$ and B s.t.

$$|\psi'(x)| \leq A|x| + B \quad \forall x \in \mathbb{R}$$

$$|\psi'(v_\infty)|^2 \leq \underbrace{(A|v_\infty| + B)^2}_{\text{integrable if } v_\infty \in L^2}$$

What about NBC (it seems they do not make sense in H^1)

$$\min \left\{ \int_a^b \dot{u}^2 + \arctan u : u(a) = A \right\}$$

As before, we can apply the direct method in H^1

Main points : $\rightarrow$ DBC controls $|u_n(x_n)|$ with $x_n = a$
 $\rightarrow$ if $F(u_n) \leq M$, then

$$\|\dot{u}_n\|_2^2 \leq M - \int_a^b \arctan u_n \leq M + \frac{\pi}{2}(b-a)$$

and this controls oscillations

After existence in H^1 , we set $\psi(t) := F(u + tr)$ and we find that

$$\int_a^b 2\dot{u} \dot{v} + \frac{1}{1+u^2} v$$

As before $\dot{u} \in H^1$ and $(2\dot{u})' = \frac{1}{1+u^2} = 2\ddot{u}$

Since we already know that $\dot{u} \in H^1$ (hence continuous) then the NBC $\dot{u}(b) = 0$ makes sense.

(NBC originates as usual after integration by parts if we consider v with $v(b) \neq 0$)

How to obtain a bound on $\|\dot{u}\|_2$?

$$F(u) = \int_a^b \varphi(\dot{u}) + \psi(x, u)$$

Want : $F(u) \leq M \Rightarrow \|\dot{u}\|_2 \leq N$

Answer : quadratic growth on φ and boundedness from below of ψ

$\varphi(x) \geq Ax^2 - B$ for some $A > 0$, $B \in \mathbb{R}$

$\psi(x, s) \geq -C$ for some $C \in \mathbb{R}$

$$M \geq F(u) \geq \int_a^b A \dot{u}^2 - B - C \quad \leadsto \text{conclusion}$$

Calculus of Variations – LECTURE 17

Note Title

26/10/2016

Variational approach to boundary value problems

Reverse engineering: given a differential equation with BC, use variational tools in order to prove existence and/or uniqueness.

Example
$$\begin{cases} u'' = \arctan u \\ u(0) = 1 \\ u(2016) = 0 \end{cases} \quad] \text{ DBC}$$

Goal: discussing existence, uniqueness and regularity

Technique: finding a min pbn. whose Euler equation is the given one.

$$(L_p)' = L_s \quad L_p = \dot{u} \quad L_s = \arctan u$$

Let us take $g: \mathbb{R} \rightarrow \mathbb{R}$ such that $g'(u) = \arctan u$ and for example $g(0) = 0$, and consider

$$\min \left\{ \int_0^{2016} \left[\frac{1}{2} \dot{u}^2 + g(u) \right] dx : u(0) = 1, u(2016) = 0 \right\}$$

$$L(x, s, p) := \frac{1}{2} p^2 + g(s)$$

We apply the direct method in order to obtain existence.



① Weak formulation in $H^1((0, 2016))$

$$F(u) = \int_0^{2016} \frac{1}{2} \dot{u}^2 + \underbrace{g(u)}_{\geq 0} \quad \text{if } u \in H^1((0, 2016)) \text{ and } u \text{ satisfies DBC}$$

② Compactness (of sublevels of F)

$$F(u) \leq M \Rightarrow \frac{1}{2} \int_0^{2016} |\dot{u}|^2 \leq M \Rightarrow \|\dot{u}\|_2^2 \leq 2M$$

If $\{u_n\}$ is any sequence with $F(u_n) \leq M$, then

$$\|\dot{u}_n\|^2 \leq \sqrt{2M} \quad \text{DBC} \Rightarrow |u_n(x)| \leq M, \quad \forall x \in [0, 2016] \quad \forall n \in \mathbb{N}$$

↑
one D condition
is enough

In the usual way, up to subsequences

$$u_n \rightarrow u_\infty \text{ unif.} \quad \dot{u}_n \rightharpoonup \dot{u}_\infty \text{ weakly in } L^2$$

③ Lower semicontinuity Assume that $u_n \rightarrow u_\infty$ as above.
Then

$$\int_0^{2016} g(u_n) \rightarrow \int_0^{2016} g(u_\infty) \quad (\text{due to unif. convergence})$$

$$\liminf_{n \rightarrow \infty} \int_0^{2016} \dot{u}_n^2 \geq \int_0^{2016} \dot{u}_\infty^2$$

$\Rightarrow F$ is LSC with respect to the notion of conv.

① + ② + ③ $\Rightarrow$ the minimum problem admits at least one solution in $H^1((0, 2016))$.

Remark Boundary cond. are stable wrt the convergence.

4 Regularity and Euler equation

$$\psi(t) := F(u+tv) \quad v \in C_c^\infty((0,2016))$$

$$\psi'(t) = \frac{d}{dt} \int_0^{2016} \frac{1}{2} \dot{u}^2 + \frac{1}{2} t^2 \dot{v}^2 + t \dot{u} \dot{v} + g(u+tv)$$

$$= \int_0^{2016} \underbrace{t \dot{v}^2 + \dot{u} \dot{v} + \arctan(u+tv) v}_{\varphi(x,t)}$$

We can exchange integral/derivative provided that $\varphi(x,t)$ is dominated by some integrable function for $t \in (-\delta, \delta)$. This follows from $\dot{u} \in L^2$ and v smooth and $\arctan$ bounded.

$$\psi'(0) = \int_0^{2016} \dot{u} \dot{v} + \arctan u \cdot v = 0 \quad \text{if } u \text{ is any min. point}$$

This can be rewritten as

$$\int_0^{2016} \underbrace{\dot{u} \dot{v}}_{\neq} = - \int_0^{2016} \underbrace{\arctan u}_{g} \cdot v \quad \forall v \in \dots$$

Equivalent to saying that g is the weak deriv. of $\neq$. In particular

$$\dot{u} \in H^1 \text{ and hence } u \in H^2$$

and

$$\ddot{u} = \arctan u$$

↑
weak second derivative

Usual BOOTSTRAP argument $\Rightarrow u$ is of class C^∞ and satisfies BCs.

Uniqueness In this case the solution to the original pbm is unique

1st approach : variational

Two steps : 1 - Any solution to the initial BVP is a minimum point for $F(u)$
 2 - $F(u)$ admits a unique min. point.

[2] is easy because $L(s,p)$ is STRICTLY convex in the pair (s,p)

[1] Again convexity. More precisely : every solution to BVP satisfies the first int. form of Euler Equ

$$\int_0^{2016} [\ddot{u} \dot{v} + \arctan u \cdot v] dx = 0$$

This implies that $u \in C^\infty$ and because of convexity it is a min. point (see lectures about indirect method)

Remark We actually proved that any solution in H^2 to the BVP is actually C^∞ .

2nd approach : ODE argument

Assume that u and v are two solutions, and set $w := u - v$

$$\ddot{u} = \arctan u$$

$$\ddot{v} = \arctan v$$

$$\ddot{w} = \arctan u - \arctan v$$

Let us multiply by $u - v$:

$\arctan x$ is increasing

$$w \ddot{w} = (\arctan u - \arctan v)(u - v) \stackrel{\downarrow}{\geq} 0$$

So $w \ddot{w} \geq 0$ for every $x \in [0, 2016]$, and therefore

$$0 \leq \int_0^{2016} w \ddot{w} dx = \underbrace{[w \dot{w}]_0^{2016}}_{=0} - \int_0^{2016} \dot{w}^2$$

because $w=0$ at the boundary

$$\Rightarrow \int_0^{2016} \dot{w}^2 = 0 \Rightarrow \dot{w} \equiv 0 \Rightarrow w \equiv 0$$

$\uparrow$
 BC

— 0 — 0 —

Example 2
$$\begin{cases} u'' = \arctan u \\ u(0) = 3 & \text{DBC} \\ \dot{u}(2016) = 0 & \text{NBC} \end{cases}$$

As before with $\min \{ F(u) : u(0) = 3 \}$.

At the end we obtain $\dot{u}(2016)$ as a by-product of Euler equ.

Example 3
$$\begin{cases} \ddot{u} = \arctan u \\ \dot{u}(0) = 0 \\ \dot{u}(2016) = 0 \end{cases} \quad \begin{aligned} &\leadsto \min F(u) \text{ without further} \\ &\quad \text{conditions} \\ &\leadsto \text{in this case } u \equiv 0 \text{ is the} \\ &\quad \text{UNIQUE solution} \end{aligned}$$

Example 4
$$\begin{cases} \ddot{u} = \arctan u \\ \dot{u}(0) = 3 \\ \dot{u}(2016) = 3 \end{cases}$$

Two possible approaches

① Variable change : $u(x) = 3x + v(x)$. What is the equ for v ?

$$\begin{cases} \ddot{v} = \arctan(v+3x) \\ \dot{v}(0) = 0 \\ \dot{v}(2016) = 0 \end{cases}$$

$$\min \left\{ \int_0^{2016} \left[\frac{1}{2} \dot{v}^2 + g(v+3x) \right] dx \right\}$$

Direct method $\leadsto$ solution $\leadsto$ uniqueness and regularity

2 Modify the Lagrangian

$$(L_p)' = L_s \quad L_p = 0 \text{ at the boundary}$$

$$L(x, s, p) = \frac{1}{2} p^2 - 3p + g(s)$$

$$\text{Euler equation becomes} \quad (\ddot{u} - 3)' = \arctan u$$

$$\ddot{u} = \arctan u$$

$$\text{BCs generated by min. procedure} \quad L_p = 0, \quad \ddot{u} - 3 = 0$$

$$\ddot{u} = 3 \text{ at boundary.}$$

Remark Compactness is ok because L has quadratic growth in p .

— 0 — 0 —

Calculus of Variations – LECTURE 18

Note Title

26/10/2016

Comments about uniqueness

$$\ddot{u} = f(u) + BC$$

If f is strictly increasing $\leadsto$ existence and uniqueness through var. approach

$$\begin{cases} \ddot{u} = -\lambda \sin u \\ u(0) = 0 \\ u(1) = 0 \end{cases} \quad \text{Easy: } u(x) \equiv 0 \text{ is a solution for every } \lambda \in \mathbb{R}$$

Claim: if λ is large enough, then the solution is NOT unique.

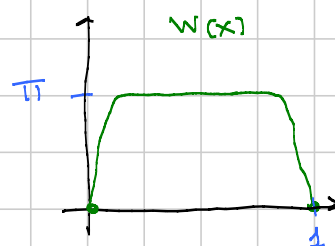
Strategy: consider $\min \left\{ \underbrace{\int_0^1 \frac{1}{2} \dot{u}^2 + \lambda \cos u}_{F(u)} : u(0)=0=u(1) \right\}$

$F(0) = \lambda$. I claim that λ is not the minimum of $F(u)$ if λ is large enough

$$F(w) = \int_0^1 \frac{1}{2} \dot{w}^2 + \lambda \cos(w) \stackrel{?}{<} \lambda$$

$$\int_0^1 \frac{1}{2} \dot{w}^2 \stackrel{?}{<} \lambda \int_0^1 (1 - \cos w) dx$$

For a given w , this is a strictly positive number



Thus for λ large $F(w) < F(0) \Rightarrow u(x) \equiv 0$ is not the minimum.

Direct method $\Rightarrow$ min point exists and solves the initial BVP.

Remark $u(x) \equiv 0$ solves (EE) but it is not the global minimum.

Example $\ddot{u} = u^3 + \cos t$

Claim: existence of at least one periodic solution

Remark: RHS is superlinear (cubic growth) $\Rightarrow$ there is no guarantee that a global solution exists.

Strategy Consider

$$F(u) := \int_0^{2\pi} \left[\frac{1}{2} \dot{u}^2 + \frac{1}{4} u^4 + \cos x \cdot u \right] dx$$

and consider

$$\min \{ F(u) : u(0) = u(2\pi) \}$$

Claim: the min pbm has a (unique) solution of class C^∞ which satisfies the equ. with

$$u(0) = u(2\pi) \quad \leftarrow \text{given}$$

$$\dot{u}(0) = \dot{u}(2\pi) \quad \leftarrow \text{generated}$$

Thus if we extend u by periodicity we have solved the initial pbm.

Easy part: every min point satisfies EE + 2BC
(usual $F(u+tv) = \dots$)

Almost easy part : existence through direct method.

Weak formulation : standard

LSC : standard if $\ddot{u}_n \rightarrow \ddot{u}_\infty$ and $u_n \rightarrow u_\infty$ unif.

Regularity + uniqueness : standard.

Remark Uniqueness to the min. pbm. $\nRightarrow$ unique of per. solution

Proof. $\cos t = \ddot{u} - u^3$ if u is T -periodic, then $\cos t$ is T -periodic

$$\Rightarrow T = 2\pi \cdot k \text{ for some } k \geq 1, k \in \mathbb{N}$$

As far as we know, a solution with min. period $= 4\pi$ might exist.

Compactness Obtaining bounds on u and $\ddot{u}$ from bounds on $F(u)$

$$M \geq F(u) = \int_0^{2\pi} \frac{1}{2} \ddot{u}^2 + g(x, u) dx \geq \int_0^{2\pi} \frac{1}{2} \ddot{u}^2 + C$$

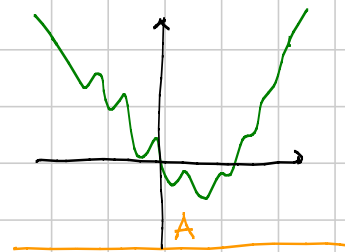
If $g(x, u)$ is bounded from below, then we obtain a bound on $\|\ddot{u}\|_2$.

In our case

$$g(x, u) = \frac{1}{4} u^4 + \cos x \cdot u$$

$$\geq \frac{1}{4} u^4 - |u| \geq A$$

$\leadsto$ bound on $\|\ddot{u}\|_2$



In order to obtain a bound on u , we observe that

$$\frac{1}{4} u^4 + \cos x \cdot u \geq \frac{1}{40} u^4 - A \quad \forall u \in \mathbb{R}$$

For a suitable A (trivial calculus as before). So

$$M \geq F(u) \geq \int_0^{2\pi} \frac{1}{40} u^4 - A$$

$$\Rightarrow \int_0^{2\pi} u^4 \leq 40M + 40A \cdot 2\pi$$

Integral bound on $u^4 \Rightarrow$ bound on $u^4(x_n)$
 $\Rightarrow$ unif. bound on $u(x)$.
 — o — o —

General discussion about compactness

$$L(x, s, p) := \varphi(p) + \psi(x, s)$$

$$\boxed{\text{Case 1}} \quad \left. \begin{array}{l} \psi(x, s) \geq -A \\ \varphi(p) \geq \underset{\substack{\uparrow \\ \text{positive}}}{B} p^2 - C \end{array} \right\} \Rightarrow \text{bound on } \|\ddot{u}\|_2$$

$$M \geq F(u) = \int_a^b \varphi(u) + \psi(x, u) \geq B \int_a^b \dot{u}^2 - (A+C)(b-a)$$

$$\Rightarrow \|\ddot{u}\|_2^2 \leq \frac{M + (A+C)(b-a)}{B}$$

$$\boxed{\text{Case 2}} \quad \left. \begin{array}{l} \varphi(p) = p^2 \\ \psi(x, s) \geq A - B|s| \\ \text{DBC} \end{array} \right\} \stackrel{?}{\Rightarrow} \text{bound on } \|\ddot{u}\|_2$$

$$\boxed{\text{YES}} \quad \begin{aligned} |u(x)| &\leq |u(a)| + |u(x) - u(a)| \\ &\leq |u(a)| + \underbrace{\|\ddot{u}\|_2 |b-a|^{1/2}}_{\frac{1}{2} - \text{H\"older continuity}} \end{aligned}$$

$$\Rightarrow \int_a^b |u(x)| \leq |u(a)|(b-a) + \|\dot{u}\|_2 |b-a|^{\frac{1}{2}} \cdot (b-a)$$

$$= C_1 + C_2 \|\dot{u}\|_2$$

Therefore

$$\begin{aligned} F(u) &= \int_a^b \dot{u}^2 + \psi(x, u) \geq \int_a^b \dot{u}^2 + A - B|u| \\ &= \|\dot{u}\|_2^2 + A(b-a) - B \int_a^b |u| \\ &\geq \|\dot{u}\|_2^2 + A(b-a) - B(C_1 + C_2 \|\dot{u}\|_2) \\ &\geq \underbrace{\|\dot{u}\|_2^2}_{\text{quadratic}} - C_3 - C_4 \underbrace{\|\dot{u}\|_2}_{\text{linear}} \end{aligned}$$

$$F(u) \leq M \Rightarrow \underbrace{\|\dot{u}\|_2^2 - C_4 \|\dot{u}\|_2 - C_3}_{\substack{\text{second order pol.} \\ \text{with pos. coeff. of leading term.}}} \leq M \Rightarrow \|\dot{u}\|_2 \leq \text{const.}$$

Case 3 Analogous argument work if

$$\varphi(p) \geq Ap^2 - B$$

$$\psi(x, s) \geq -C - D|s|^a \quad \text{for some } a < 2$$

Remark 1 $F(u) = \int \dot{u}^2 - u^3$ no bound is possible

$$F(nw) = n^2 \int \dot{w}^2 - n^3 \int w^3$$

$$\leadsto \inf = -\infty$$



Remark 2 If we are as in case 3, but with $a=2$
Then constants come into play, namely the possibility
to obtain a bound depends on $A, D, (b-a)$
— o — o —

Calculus of Variations – LECTURE 19

Note Title

28/10/2016

Preliminaries about LSC functions

General setting: (X, d) metric space, $f: X \rightarrow \bar{\mathbb{R}}$ is a function

Prop. (characterization of LSC)

The following facts are equivalent

(i) For every $x_0 \in X$ which is not isolated

$$\liminf_{y \rightarrow x} f(y) \geq f(x_0),$$

(ii) For every conv. sequence $x_n \rightarrow x_\infty$

$$\liminf_{n \rightarrow \infty} f(x_n) \geq f(x_\infty),$$

(iii) For every $M \in \mathbb{R}$ the sublevel

$$\{x \in X : f(x) \leq M\} \text{ is closed}$$

(iv) The EPIGRAPH

$$\{(x, y) \in X \times \mathbb{R} : y \geq f(x)\}$$

is closed in $X \times \mathbb{R}$.

Proof. Exercise.

— o — o —

Def. (LSC envelope). Let $f: X \rightarrow \bar{\mathbb{R}}$ be any function.
The LSC envelope is the function

$$\hat{F}(x) = \sup \{ g(x) : g: X \rightarrow \bar{\mathbb{R}} \text{ LSC and } g(y) \leq f(y) \forall y \in X \}$$

(supremum of all LSC functions below f)

Prop. $\hat{F}$ is LSC. More generally: the supremum of any family of LSC functions is LSC.

Proof. Let $f_i: X \rightarrow \bar{\mathbb{R}}$ for every $i \in I$. Let

$$f(x) := \sup \{ f_i(x) : i \in I \}$$

Assume f_i is LSC for every $i \in I$.

Two ways

- with sublevels $\{x \in X : f(x) \leq M\} = \bigcap_{i \in I} \{x \in X : f_i(x) \leq M\}$

- the same with epigraphs.

— o — o —

Def. Given $f: X \rightarrow \bar{\mathbb{R}}$, the relaxation is the function

$$\bar{f}(x) := \inf \{ \liminf_{n \rightarrow \infty} f(x_n) : x_n \rightarrow x \}$$

— o — o —

Remark Why we do not define

$$\bar{f}(x) := \liminf_{y \rightarrow x} f(y) \quad \text{FALSE}$$

Big difference: constant sequence $x_n \equiv x$ is allowed in the defn. of $\bar{f}(x)$ but not in computing the $\liminf$

Lemma Assume $\bar{f}(x)$ is the relaxation of $f(x)$.

Then

$$\forall x \in X, \forall \varepsilon > 0, \forall r > 0 \exists y \in X \text{ such that } d(x, y) \leq r \text{ and } f(y) \leq \bar{f}(x) + \varepsilon \quad \text{if } \bar{f}(x) \in \mathbb{R}$$

Proof. By definition of infimum, there exists a sequence $x_n \rightarrow x$ such that

$$\liminf_{n \rightarrow \infty} f(x_n) \leq \bar{f}(x) + \frac{1}{2} \varepsilon$$

In particular this implies that

$$f(x_n) \leq \bar{f}(x) + \varepsilon \quad (*)$$

for infinitely many indices n . Now we can take $y := x_n$ for n large enough and among the indices for which $(*)$ holds.

— o — o —

Property 1 $\bar{f}(x) \leq f(x) \quad \forall x \in X$

Proof Trivial! consider $x_n \equiv x$

Property 2 The infimum in the defn of $\bar{f}(x)$ is actually a min, namely there exists a sequence $x_n \rightarrow x$ such that

$$\bar{f}(x) = \lim_{n \rightarrow \infty} f(x_n)$$

Proof: Let us assume that $\bar{f}(x) \in \mathbb{R}$ (other cases are simpler).
Apply the lemma with

$$\epsilon := \frac{1}{n} \quad \delta := \frac{1}{n}$$

$\leadsto$ obtain a point $x_n \in X$ s.t.

$$\underbrace{d(x, x_n) \leq \frac{1}{n}}_{x_n \rightarrow x}$$

$$\underbrace{f(x_n) \leq \bar{f}(x) + \frac{1}{n}}_{\downarrow}$$

$$\limsup_{n \rightarrow \infty} f(x_n) \leq \bar{f}(x)$$

On the other hand

$$\bar{f}(x) \leq \liminf_{n \rightarrow \infty} f(x_n) \leq \limsup_{n \rightarrow \infty} f(x_n) \leq \bar{f}(x)$$

In other words, $\underbrace{f(x_n)}_{\rightarrow} \underbrace{\bar{f}(x)}_{\leftarrow}$

Definition Let $\bar{f}(x)$ be the relaxation of $f(x)$.

Every sequence $x_n \rightarrow x$ s.t. $f(x_n) \rightarrow \bar{f}(x)$ is called a RECOVERY SEQUENCE for x .

Property 2 restated: every $x \in X$ admits a recovery sequence
 $\underbrace{\quad}_0 \quad \underbrace{\quad}_0 \quad \underbrace{\quad}_\infty$

Property 3 $\bar{f}$ is LSC.

Proof. Let us take any sequence $x_n \rightarrow x_\infty$. We need to prove that

$$\liminf_{n \rightarrow \infty} \bar{f}(x_n) \geq \bar{f}(x_\infty)$$

For every $n \in \mathbb{N}$ let us apply the lemma with base point x_n

$$\epsilon := \frac{1}{n} \quad \delta := \frac{1}{n}.$$

$\leadsto$ obtain a point $y_n \in X$ such that

$$d(y_n, x_n) \leq \frac{1}{n}$$



$$y_n \rightarrow x_\infty$$

$$f(y_n) \leq \bar{f}(x_n) + \frac{1}{n}$$



$$\liminf_{n \rightarrow \infty} f(y_n) \leq \liminf_{n \rightarrow \infty} \bar{f}(x_n)$$

We conclude by remarking that $\bar{f}(x_\infty) \leq \liminf_{n \rightarrow \infty} f(y_n)$.

— o — o —

Property 4 The relaxation of f coincides with the LSC envelope of f .

Proof. $\bar{f} := \text{relaxation}$ $\hat{f} : \text{LSC envelope}$

$$\bar{f} \leq \hat{f}$$

Trivial, because we know that $\bar{f}$ is LSC and below f , so it competes in the defn. of $\hat{f}$.

$$\bar{f} \geq \hat{f}$$

We need that every $g: X \rightarrow \bar{\mathbb{R}}$ which is LSC and below f is actually below $\bar{f}$. Let us take any such g , and any $x \in X$.

Let $x_n \rightarrow x$ be any recovery sequence for x

$$g(x) \leq \liminf_{n \rightarrow \infty} g(x_n) \leq \liminf_{n \rightarrow \infty} f(x_n) = \bar{f}(x)$$

$\uparrow$ $g \text{ is LSC}$ $\uparrow$ $g \leq f$ $\uparrow$ $x_n \text{ is a recovery.}$

— o — o —

Property 5 (Stability under continuous perturbations)

Assume $f: X \rightarrow \bar{\mathbb{R}}$ and let $g: X \rightarrow \bar{\mathbb{R}}$ continuous. Then

$$\overline{f+g} = \bar{f} + g$$

Proof. Let $x_n \rightarrow x$ be any sequence.

$$\begin{aligned} \liminf_{n \rightarrow \infty} (f(x_n) + g(x_n)) &\geq \liminf_{n \rightarrow \infty} f(x_n) + \liminf_{n \rightarrow \infty} g(x_n) \\ &\geq \bar{f}(x) + g(x) \end{aligned}$$

This proves that $\overline{f+g}(x) \geq \bar{f}(x) + g(x)$ (requires just that g is LSC)

Now let $x_n \rightarrow x$ be a recovery seq. for x for f . Then

$$\begin{aligned} \liminf_{n \rightarrow \infty} (f(x_n) + g(x_n)) &\leq \limsup_{n \rightarrow \infty} (f(x_n) + g(x_n)) \\ &\leq \limsup_{n \rightarrow \infty} f(x_n) + \limsup_{n \rightarrow \infty} g(x_n) \\ &\leq \bar{f}(x) + g(x) \quad \text{(enough that } g \text{ is usc)} \end{aligned}$$

This proves that $\bar{f}(x) + g(x) \geq \overline{f+g}(x)$, and actually that

$x_n \rightarrow x$ is a recovery also for the sum.

Calculus of Variations – LECTURE 20

Note Title

28/10/2016

RELAXATION AND MIN. PBMS

General setting: (X, d) metric space, $f: X \rightarrow \bar{\mathbb{R}}$, $\bar{f}: X \rightarrow \bar{\mathbb{R}}$
relaxation of f .

Property 1 $\inf \{f(x) : x \in X\} = \inf \{\bar{f}(x) : x \in X\}$

Proof. Assume $\bar{I}$ is the 2nd inf and $\bar{I} \in \mathbb{R}$ (the other case is simpler).

By defn. of inf, there exists $x \in X$ s.t.

$$\bar{f}(x) \leq \bar{I} + \varepsilon$$

By the usual lemma, there exists a point $y \in X$ (actually as close as we want to x) such that

$$f(y) \leq \bar{f}(x) + \varepsilon \leq \bar{I} + 2\varepsilon$$

This implies that $\inf_{x \in X} f \leq \inf_{x \in X} \bar{f}$

The reverse inequality is trivial because $\bar{f} \leq f$.

Defn We say that f is coercive if there exists a compact set $K \subset X$ s.t.

$$\inf_{x \in K} f(x) = \inf_{x \in X} f(x)$$

Property 2 Assume that f is coercive.

Then

$$\min \{ \bar{f}(x) : x \in X \} \quad (= \inf \{ f(x) : x \in X \})$$

exists.

Proof. We already know that $\bar{f}$ is LSC, and hence it admits the minimum in K . We claim that the min in K is actually also the min in X . (see end of file)

Let I denote the inf of f in K (or in X).

Then there exists a sequence $\{x_n\} \subseteq K$ such that

$$f(x_n) \rightarrow I$$

Up to subsequences we can assume that $x_n \rightarrow x_\infty$ (because K is compact). Claim: x_∞ is a min. point of $\bar{f}(x)$.

Let $y \in X$ be any point, and let $y_n \rightarrow y$ be a recov. seq. for y . Then

$$\begin{array}{ccccccc} \bar{f}(y) = \lim_{n \rightarrow \infty} f(y_n) & \geq I & = \lim_{n \rightarrow \infty} f(x_n) & \geq \bar{f}(x_\infty) \\ \uparrow \text{ } y_n \text{ is recovery} & \uparrow I = \inf_{x \in X} & \downarrow \text{defn. of } x_n & \uparrow \text{defn. of } \bar{f} \end{array}$$

Property 3 Assume again that f is coercive. Let $\{x_n\} \subseteq K$ be any sequence such that

$$f(x_n) \rightarrow \inf_{x \in X} f(x) \quad (\text{min. sequence})$$

Then any limit point of x_n is a min point of the relaxation $\bar{f}$.

Proof : see above

Remark If there exists $M > I$ such that

$$\{x \in X : f(x) \leq M\}$$

is compact, then any minimizing sequence is relatively compact (it lies in the sublevel) and all its limit points are min. points of $\bar{f}(x)$.

Property 4 Assume as in the remark and in addition assume that $\bar{f}$ has a UNIQUE min. point x_0 . Then the set of min. seq. for f coincides with the set of recovery sequence for x_0 .

$$\begin{array}{ccccc} \text{---} & \circ & \text{---} & \circ & \text{---} \\ \text{---} & \circ & \text{---} & \circ & \text{---} \end{array}$$

Defn. Let $f: X \rightarrow \bar{\mathbb{R}}$ be any function.

A subset $D \subseteq X$ is called DENSE IN ENERGY (wrt f) if

$$\forall x \in X \exists \{x_m\} \subseteq D \text{ s.t. } x_m \rightarrow x \text{ and } f(x_m) \rightarrow f(x)$$

Prop D is dense in energy iff

$$\{(d, f(d)) : d \in D\}$$

is dense in the graph (wrt to $X \times \mathbb{R}$)

Example If f is cont., any $D \subseteq X$ dense is dense in energy.

Without continuity this is false.

Lemma Let $\varphi: X \rightarrow \mathbb{R}$ and $\psi: X \rightarrow \mathbb{R}$ be two functions
Let $D \subseteq X$.

Let us assume that

- (i) D is dense in energy for ψ ,
- (ii) φ is LSC,
- (iii) $\varphi(x) \leq \psi(x)$ for every $x \in D$.

Then

$$\varphi(x) \leq \psi(x) \quad \forall x \in D.$$

Proof. Let $x \in X$ be any point. Let $\{x_n\} \subseteq D$ be as in defn. of the subset dense in energy.
Then

$$\varphi(x) \leq \liminf \varphi(x_n) \leq \liminf \psi(x_n) = \psi(x)$$

$\uparrow$ φ is LSC $\uparrow$ (iii) $\uparrow$ D is dense in energy for ψ .

— o — o —

Question: assume that $f: X \rightarrow \overline{\mathbb{R}}$ is given.

Assume that we suspect that $g: X \rightarrow \overline{\mathbb{R}}$ is the relaxation of f .

How to prove it?

First strategy Need to check 3 things

- 1 $f(x) \geq g(x)$ for every $x \in X$
- 2 $g(x)$ is LSC
- 3 there exists $D \subseteq X$ which is dense in energy for g such that

$$\forall x \in D \quad \exists x_n \rightarrow x \text{ s.t. } f(x_n) \rightarrow g(x)$$

(existence of a recovery for every $x \in D$)

Proof $\bar{f}(x) \geq g(x)$ Trivial because $\bar{f} = \hat{f} \geq g$

$\bar{f}(x) \leq g(x)$ Due to $\textcircled{3}$ we know that

$$\bar{f}(x) \leq \liminf_{n \rightarrow \infty} f(x_n) = g(x) \quad \forall x \in D$$

$\uparrow$ defn of $\bar{f}$ $\uparrow$ $\textcircled{3}$

Now we apply the lemma with

$$\varphi(x) := \bar{f}(x)$$

$\uparrow$
LSC because
it's a relaxation

$$\psi(x) := g(x)$$

$\uparrow$
admits a dense in energy
where $\varphi \leq \psi$

$$\Rightarrow \bar{f}(x) \leq g(x) \text{ for every } x \in X.$$

— o — o —

Added during lecture 21

f coercive $\Rightarrow \bar{f}$ admits min in X

Proof. $\bar{f}$ is LSC in K . Let x_0 be a min. point of $\bar{f}$ in K .

Let $y \in X$ be any point. Let $y_n \rightarrow y$ be a recovery seq.

Then

$$\bar{f}(y) = \lim_{n \rightarrow \infty} f(y_n) \geq \inf_{x \in X} f(x) \stackrel{\text{coerciveness}}{=} \inf_{x \in K} f(x)$$

$\uparrow$ y_n is recovery $\uparrow$ $f(y_n) \geq \inf$ $f \geq \bar{f} \rightarrow$

$$\geq \inf_{x \in K} \bar{f}(x)$$

$$(\text{defn. of } x_0) = \bar{f}(x_0).$$

— o — o —

Calculus of Variations - LECTURE 21

Note Title

02/11/2016

EXTENSION BY RELAXATION

Given: set X and a function $f: X \rightarrow \bar{\mathbb{R}}$
 metric space $\hat{X}$ containing X as a subset.

Goal: extending f to a function $\hat{f}: \hat{X} \rightarrow \bar{\mathbb{R}}$

General procedure: consider $f_1: \hat{X} \rightarrow \mathbb{R}$

$$f_1(x) = \begin{cases} f(x) & x \in X \\ +\infty & \text{otherwise} \end{cases}$$

Take $\hat{f}$ to be the relaxation of f_1 .

We have for free that

1 - $\hat{f}$ is LSC,

2 - $\hat{f}(x) \leq f(x)$ in X .

Main example $X := C^1([a, b])$

$$F(u) = \int_a^b (\dot{u}^2 + u^2) dx$$

Let $\hat{X} := L^2((a, b))$ or $\hat{X} := C^0([a, b])$

Problem: extending $F(u)$ to $\hat{X}$.

In both cases, the extension is

$$\hat{F}(u) = \begin{cases} \int_a^b (\dot{u}^2 + u^2) dx & \text{if } u \in H^1((a, b)) \\ +\infty & \text{if } u \in \hat{X} \setminus H^1((a, b)) \end{cases}$$

Remark This provides a 3rd defn. of the Sobolev space H^1 , as the set of all u in C^0 or L^2 such that the extension by relaxation is $< +\infty$.

— o — o —

Strategies for proving that G is the relaxation of F

Strategy 1 ① For every $x_n \rightarrow x_\infty$ it turns out that

$$(\text{LIMINF INEQ}) \quad \liminf_{n \rightarrow \infty} F(x_n) \geq G(x_\infty) \quad (\bar{F} \geq G)$$

② For every $x \in X$ there exists a sequence $x_n \rightarrow x$ such that

$$\liminf_{n \rightarrow \infty} F(x_n) \leq G(x) \quad (\bar{F} \leq G)$$

In practise, proving ② is equivalent to finding $x_n \rightarrow x$ s.t.

$$(\text{LIMSUP INEQ}) \quad \limsup_{n \rightarrow \infty} F(x_n) \leq G(x)$$

More important: it is enough to consider $x \in D$, where D is dense in energy wrt G .

Strategy 2 ① G is L.S.C.

② $G(x) \leq F(x) \quad \forall x \in X$

③ the existence of a recovery sequence for every $x \in D$ (dense in energy)

— o — o —

Proof of the example Want to find relaxation in $L^2(a,b)$

$$F(u) = \begin{cases} \int_a^b u^2 + u^2 & \text{if } u \in C^1 \\ +\infty & \text{otherwise} \end{cases}$$

Claim

$$\bar{F}(u) = G(u) = \begin{cases} \int_a^b u^2 + u^2 & \text{if } u \in H^1 \\ +\infty & \text{otherwise} \end{cases}$$

LIMINF INEQUALITY Given any $u_n \xrightarrow{L^2} u_\infty$, prove that

$$\liminf F(u_n) \geq G(u_\infty)$$

We can assume WLOG that $LHS < +\infty$. This implies that there exists u_{n_k} and M such that

$$F(u_{n_k}) \leq M \quad \forall k \in \mathbb{N}$$

$$\Rightarrow \|u_{n_k}\|_2^2 \leq M \Rightarrow \text{up to further subsequences}$$

$$u_{n_k} \rightharpoonup \dot{v} = u_\infty$$

This implies that $u_\infty \in H^1(a,b)$. At this point we can pass to the liminf as usual

$$\liminf_{k \rightarrow \infty} F(u_{n_k}) = \liminf_{k \rightarrow \infty} \int_a^b (\dot{u}_{n_k}^2 + u_{n_k}^2) \geq \int_a^b \dot{u}_\infty^2 + u_\infty^2$$

$\uparrow$
 LSC of the
 weak.

This proves the liminf ineq. up to choosing n_k in such a way that

$$\liminf_{k \rightarrow \infty} F(u_{n_k}) = \liminf_{n \rightarrow \infty} F(u_n)$$

LIMSUP INEQ. For every $u \in L^2$ we need a sequence $u_m \rightarrow u$ such that

$$\limsup_{m \rightarrow \infty} F(u_m) \leq G(u)$$

This is trivial if $u \notin H^1$ because $G = +\infty$ there. So assume that $u \in H^1$. By definition (H) of Sobolev spaces

$$\exists \{u_m\} \subseteq C^\infty \text{ st. } u_m \rightarrow u \text{ and } u_m' \rightarrow u'.$$

This is exactly what we need.

— o — o —

Remark. Proving by strategy 1 that G is the relaxation of F implies as a by-product that G is LSC.

— o — o —

Very important example (not discussed here). Let us consider a geometric problem, for example in $\mathbb{R}^2$

$$F(A) = \int_A f(x) dx + \text{Per}(A)$$



where $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ (for example continuous), $A \subseteq \mathbb{R}^2$ is a bounded subset, and $\text{Per}(A)$ is the perimeter of A .

Question: What is the perimeter?



A way to define the perimeter for non-regular sets A is the following

union of

- Define X as the set of polygons (where Per is a classical notion)
- Define $\hat{X}$ as the set of bounded measurable subsets of $\mathbb{R}^2$, which is a metric space w.r.t. to L^1 distance of charact. functions
- Extend by relaxation Per from X to $\hat{X}$.

Example

$$\inf \left\{ \underbrace{\int_a^b [\sin(u) + (u-x)^2] dx}_{F(u)} : u \in C^{2,2}([a,b]) \right\}$$

Claim: $\bar{F}(u) = \int_a^b (u-x)^2 dx - (b-a)$

If the claim is true, then $\inf F = -(b-a)$ because

$$\inf F = \inf \bar{F}$$

In turn, the claim is equivalent to showing that

$$G(u) := \int_a^b \sin(u) \quad \text{has relaxation} \quad \bar{G}(u) := \int_a^b -1 = -(b-a)$$

because the second term $\int_a^b (u-x)^2 dx$ is continuous.

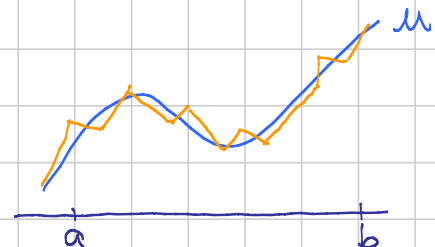
In order to prove that $\bar{G}$ is the relaxation of G , the liminf ineq. is trivial because

$$G(u_n) \geq \int_a^b (-1) \quad \text{for every } u_n.$$

It remains to prove that there exists a recovery sequence, namely every u can be approximated, for example in C^0 or L^2 , by a sequence u_n with $\sin(u_n) = -1$.

It is enough that u_n is piecewise affine and

$$u_n(x) \in \left\{ \frac{3\pi}{2} + 2k\pi : k \in \mathbb{Z} \right\}$$



for every $x \in [a,b]$ (on each subinterval)

Remark Enough to do this for every $u \in C^1$ or C^∞ or any other $\mathcal{D}$ dense in energy.

Calculus of Variations

LECTURE 22

Note Title

02/11/2016

GAMMA CONVERGENCE

Setting : (X, d) metric space
 $F_m: X \rightarrow \bar{\mathbb{R}}$

Def. The sequence F_m Γ -converges to $F_\infty: X \rightarrow \bar{\mathbb{R}}$ if the following two properties are satisfied

(i) (LIMINF INEQUALITY) for every conv. seq. $x_m \rightarrow x_\infty$ in X

$$\liminf_{m \rightarrow \infty} F_m(x_m) \geq F_\infty(x_\infty)$$

(ii) (LIMSUP INEQUALITY) For every $x \in X$, there exists $x_m \rightarrow x$ such that

$$\limsup_{m \rightarrow \infty} F_m(x_m) \leq F_\infty(x_\infty)$$

Defn Every seq. $x_m \rightarrow x$ satisfying the limsup ineq. is called RECOVERY SEQUENCE for x .

Basic fact A recovery seq. actually satisfies (if $F_m \xrightarrow{\Gamma} F_\infty$)

$$\lim_{m \rightarrow \infty} F_m(x_m) = F_\infty(x_\infty)$$

Proof.

$$F_\infty(x_\infty) \overset{(i)}{\leq} \liminf_{m \rightarrow \infty} F_m(x_m) \overset{\text{trivial}}{\leq} \limsup_{m \rightarrow \infty} F_m(x_m) \overset{(ii)}{\leq} F_\infty(x_\infty).$$

— o — o —

Property 1 Assume that $F_m(x) = F(x)$ for every $m \in \mathbb{N}$.
Then

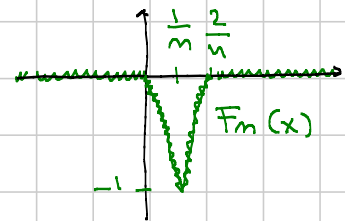
$$F_m \xrightarrow{\Gamma} \bar{F}$$

(the relaxation is the Γ -limit of the constant sequence $F_m \equiv F$)

Proof. Equivalent to strategy 1 of the verification of relaxation.

Remark This shows that Γ -convergence is not implied by pointwise convergence or uniform convergence.

Example 1 $F_m: \mathbb{R} \rightarrow \mathbb{R}$ as in the picture



Claim

$$F_m \xrightarrow{\Gamma} F_\infty(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$$

Proof. (LIMIT INEQUALITY) $\liminf F_m(x_n) \geq F_\infty(x_\infty)$
for every $x_n \rightarrow x_\infty$

Trivial if $x_\infty = 0$. If $x_\infty \neq 0$, then also $F_m(x_n) = 0$

(LIMSUP INEQ) $\forall x \in \mathbb{R} \exists x_n \rightarrow x$ s.t. $\limsup F_m(x_n) \leq F_\infty(x)$

If $x \neq 0$ then take $x_n \equiv x$

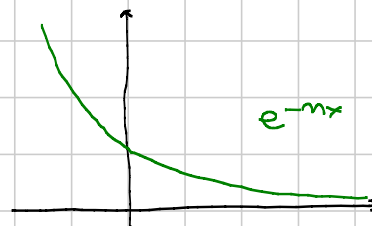
If $x = 0$ then take $x_n = \frac{1}{n}$ ← recovery seq. for $x = 0$

Remark In example 1 the pointwise limit exists and it is $\equiv 0$, and therefore different from the Γ -limit.

Example 2 $F_n: \mathbb{R} \rightarrow \mathbb{R}$ $F_n(x) = e^{-nx}$

The pointwise limit is

$$\lim_{n \rightarrow \infty} F_n(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ +\infty & \text{if } x < 0 \end{cases}$$



The Γ -limit is

$$\Gamma\text{-}\lim_{n \rightarrow \infty} F_n(x) = F_\infty(x) := \begin{cases} +\infty & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

(liminf ineq.) $\liminf F_n(x_n) \geq F_\infty(x)$

Trivial for $x \geq 0$, and easy for $x < 0$ (if $x_n \rightarrow x < 0$, then $x_n \leq -\frac{1}{2}x$ eventually)

(limsup ineq.) $\limsup F_n(x_n) \leq F_\infty(x)$

Trivial for $x \neq 0$, because we can take $x_n \equiv x$
For $x = 0$, we can take

$$x_n := \frac{1}{\sqrt{n}}$$

so that $F_n(x_n) = e^{-\sqrt{n}} \rightarrow 0$

Example 3 Assume that

- (i) $F_n(x) \rightarrow F_\infty(x)$ uniformly on compact subsets of X
- (ii) $F_\infty(x)$ is continuous in X

Then $F_n(x) \xrightarrow{\Gamma} F_\infty(x)$

Proof. (linsup inequality) Take $x_n \equiv x$ (pointwise convergence is enough)

(liminf inequality) For every $x_n \rightarrow x_\infty$ assumptions (i) and (ii) guarantee that

$$F_n(x_n) \longrightarrow F_\infty(x_\infty)$$

$$\left[\underbrace{d(F_\infty(x_\infty), F_n(x_n))}_{\rightarrow 0} \leq \underbrace{d(F_\infty(x_\infty), F_\infty(x_n))}_{\rightarrow 0} + \underbrace{d(F_\infty(x_n), F_n(x_n))}_{\rightarrow 0} \right]$$

Property Assume that $F_n \xrightarrow{\Gamma} F_\infty$
 Assume that $G: X \rightarrow \bar{\mathbb{R}}$ is continuous

Then $F_n + G \xrightarrow{\Gamma} F_\infty + G$ (stability under const. perturbations)

Proof. (liminf ineq.) Take any $x_n \rightarrow x_\infty$. Then

$$\begin{aligned} \liminf_{n \rightarrow +\infty} F_n(x_n) + G(x_n) &\geq \underbrace{\liminf_{n \rightarrow +\infty} F_n(x_n)}_{\Gamma\text{-conv.}} + \underbrace{\liminf_{n \rightarrow +\infty} G(x_n)}_{G \text{ is LSC}} \\ &\geq F_\infty(x_\infty) + G(x_\infty) \end{aligned}$$

(linsup ineq.) Let $x \in X$ and let $x_n \rightarrow x$ be a rec. seq. for F_n .
 Then

$$\begin{aligned} \limsup F_n(x_n) + G(x_n) &\leq \limsup F_n(x_n) + \limsup G(x_n) \\ &\leq \underbrace{F_\infty(x_\infty)}_{\Gamma\text{-conv.}} + \underbrace{G(x_\infty)}_{G \text{ is USC}} \end{aligned}$$

Remark It can be generalized to $F_n \xrightarrow{\Gamma} F_\infty$ and $G_n \rightarrow G_\infty$ unif with G_∞ continuous.

Proof: exercise.

Def. Given $F_n: X \rightarrow \bar{\mathbb{R}}$. One defines

$$\Gamma\text{-}\liminf F_n(x) := \inf \left\{ \liminf_{n \rightarrow \infty} F_n(x_n) : x_n \rightarrow x \right\}$$

$$\Gamma\text{-}\limsup F_n(x) := \inf \left\{ \limsup_{n \rightarrow \infty} F_n(x_n) : x_n \rightarrow x \right\}$$

Proposition • $\Gamma\text{-}\liminf F_n(x) \leq \Gamma\text{-}\limsup F_n(x) \quad \forall x \in \mathbb{R}$

- we have equality for every $x \in X$ iff the Γ -limit exists.

Proof. Trivial from the definitions
— o — o —

The other side of the moon. What we defined are actually

$\Gamma^-\text{-}\liminf$ and $\Gamma^-\text{-}\limsup$.

Analogous definitions can be made with $\sup$ instead of $\inf$.
This leads to

$$\Gamma^+\text{-}\liminf F_n(x) := \sup \left\{ \liminf_{n \rightarrow \infty} F_n(x_n) : x_n \rightarrow x \right\}$$

$$\Gamma^+\text{-}\limsup F_n(x) := \sup \left\{ \limsup_{n \rightarrow \infty} F_n(x_n) : x_n \rightarrow x \right\}$$

Comment Γ^- is suited for min pblms
 Γ^+ is suited for max pblms.
— o — o —

Calculus of Variations — LECTURE 23

Note Title

04/11/2016

(X, d) metric space, $F_n: X \rightarrow \bar{\mathbb{R}}$

$$\Gamma\text{-}\liminf_{n \rightarrow \infty} F_n(x) := \inf \left\{ \liminf_{m \rightarrow \infty} F_m(x_m) : x_m \rightarrow x \right\}$$

$$\Gamma\text{-}\limsup_{n \rightarrow \infty} F_n(x) := \inf \left\{ \limsup_{m \rightarrow \infty} F_m(x_m) : x_m \rightarrow x \right\}$$

If $\Gamma\text{-}\liminf F_n(x) = \Gamma\text{-}\limsup F_n(x)$ for every x , the common value is the Γ -limit.

- Goals :
- ① Γ -liminf and Γ -limsup are LSC in X
 - ② Convergence of minima and minimizers

Lemma 1 (Gamma convergence and subsequences)

Let n_k be an increasing sequence of integers.

Then

$$\begin{aligned} \Gamma\text{-}\liminf_{n \rightarrow \infty} F_n(x) &\leq \Gamma\text{-}\liminf_{k \rightarrow \infty} F_{n_k}(x) \leq \Gamma\text{-}\limsup_{k \rightarrow \infty} F_{n_k}(x) \\ &\leq \Gamma\text{-}\limsup_{n \rightarrow \infty} F_n(x) \end{aligned}$$

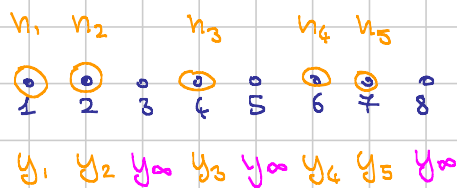
In particular, if $\Gamma\text{-}\lim_{n \rightarrow \infty} F_n(x)$ exists, then also $\Gamma\text{-}\lim_{k \rightarrow \infty} F_{n_k}(x)$ exists.

Proof. Let $y_k \rightarrow y_\infty$ be any sequence. Let F_∞ denote the Γ -lim of $F_n(x)$. Claim

$$\liminf_{k \rightarrow \infty} F_{n_k}(y_k) \geq F_\infty(y_\infty)$$

Let us consider the sequence x_n defined by setting

$$x_i = \begin{cases} y_k & \text{if } i = n_k \text{ for some } k \\ y_\infty & \text{otherwise} \end{cases}$$



Clearly $x_n \rightarrow y_\infty$, and therefore

$$\begin{aligned} F_\infty(y_\infty) &\leq \liminf_{n \rightarrow \infty} F_n(x_n) \leq \liminf_{k \rightarrow \infty} F_{n_k}(x_{n_k}) \\ &= \liminf_{k \rightarrow \infty} F_{n_k}(y_k) \end{aligned}$$

This proves the first inequality. The last one is analogous.

Lemma 2 (For T -liminf)

Assume that $F_\infty(x) = T\text{-}\liminf_{n \rightarrow \infty} F_n(x)$.

Then

$$\forall y \in X, \forall i \in \mathbb{N}, \forall \varepsilon > 0, \forall \rho > 0 \quad \exists m \in \mathbb{N} \quad \exists z \in X \text{ s.t.}$$

$$m \geq i, \quad d(y, z) \leq \rho, \quad F_m(z) \leq F_\infty(y) + \varepsilon$$

Proof. By definition of T -liminf there exists $y_m \rightarrow y$ such that

$$\liminf_{n \rightarrow \infty} F_n(y_m) \leq F_\infty(y) + \frac{1}{2} \varepsilon$$

In particular

$$F_m(y_m) \leq F_\infty(y) + \varepsilon \quad \text{for infinitely many } m\text{'s}$$

Now it is enough to take $m :=$ index large enough for which the relation is true and $z := y_i$
 $\text{---} \circ \text{---} \circ \text{---}$ $\uparrow m, \text{ of course (added after video)}$

Remark Lemma 2 can be extended more easily to the Γ -limsup.

Lemma 3 (Convergence of minima)

Assume F_∞ is the Γ -liminf.

Then for every compact set $K \subseteq X$ it turns out that

$$\liminf_{n \rightarrow \infty} \inf_{x \in K} F_n(x) \geq \inf_{x \in K} F_\infty(x)$$

Proof. By definition of $\inf$, for every $m \in \mathbb{N}$ there exists $x_m \in K$ such that

$$F_n(x_m) \leq \underbrace{\inf_{x \in K} F_n(x)}_{I_n} + \frac{1}{n}$$

Let us take a subsequence n_k such that

$$\liminf_{n \rightarrow \infty} I_n = \lim_{k \rightarrow \infty} I_{n_k} \quad \text{and} \quad x_{n_k} \rightarrow x_\infty \quad (K \text{ is compact})$$

Then

$$\begin{aligned} \liminf_{n \rightarrow \infty} I_n &= \lim_{k \rightarrow \infty} I_{n_k} \geq \liminf_{k \rightarrow \infty} I_{n_k} \\ &\geq \liminf_{k \rightarrow \infty} \left(F_{n_k}(x_{n_k}) - \frac{1}{n_k} \right) \\ &= \liminf_{k \rightarrow \infty} F_{n_k}(x_{n_k}) \end{aligned}$$

$$\text{Lemma 1} \quad \rightarrow \geq F_\infty(x_\infty) \geq \inf_{x \in K} F_\infty(x).$$

$\text{---} \circ \text{---} \circ \text{---}$

Theorem 1 Let $F_\infty(x)$ be the Γ -liminf.
Then F_∞ is LSC.

Proof. Let $x_n \rightarrow x_\infty$ be any sequence. We claim that

$$\liminf_{n \rightarrow \infty} F_\infty(x_n) \geq F_\infty(x_\infty)$$

We define a sequence n_k and a sequence y_k as follows:

- select n_1 and y_1 at random
- assume that $n_1, \dots, n_{k-1}$ and $y_1, \dots, y_{k-1}$ have already been selected. Apply Lemma 2 with

$$y := x_k, \quad i := n_{k-1} + 1, \quad \varepsilon := \frac{1}{k}, \quad \eta := \frac{1}{k}$$

We obtain z , which we call y_k , and n , which we call n_k , such that

$$\underbrace{n_k \geq n_{k-1} + 1}_{\{n_k\} \text{ is increasing}}, \quad \underbrace{d(x_k, y_k) \leq \frac{1}{k}}_{y_k \rightarrow x_\infty}, \quad \underbrace{F_\infty(y_k) \leq F_{n_k}(x_k) + \frac{1}{k}}_{\substack{\text{swap indices} \\ F_{n_k}(y_k) \leq F_\infty(x_k) + \frac{1}{k} \\ \text{(added after video)}}$$

In conclusion $y_k \rightarrow y_\infty$ and

$$\liminf_{k \rightarrow \infty} F_\infty(y_k) \leq \liminf_{k \rightarrow \infty} F_{n_k}(x_k) \xrightarrow{\text{swap indices}} \liminf_{k \rightarrow \infty} F_\infty(x_k) \xrightarrow{\text{Lemma 1}} F_\infty(x_\infty) \quad \text{(added after video)}$$

which proves that F_∞ is LSC. $\square$

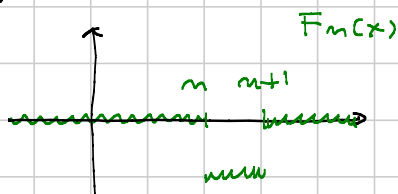
The proof that Γ -limsup is LSC is analogous starting from the second version of Lemma 2.

Convergence of minima and minimizers

Assume that $F_n \xrightarrow{\Gamma} F_\infty$. In general it is not true that

$$\inf_{x \in X} F_n(x) \rightarrow \inf_{x \in X} F_\infty(x)$$

Example $X = \mathbb{R}$



$$\inf F_n = -1, \quad \inf F_\infty = 0.$$

Definition The sequence $F_n: X \rightarrow \overline{\mathbb{R}}$ is **EQUICOERCIVE** if there exists a compact set $K \subseteq X$ such that

$$\inf_{x \in X} F_n(x) = \inf_{x \in K} F_n(x) \quad \forall n \in \mathbb{N}.$$

Theorem Assume that $F_n \xrightarrow{\Gamma} F_\infty$ and F_n is equicoercive. Then

(1) F_∞ has a minimum in X

(2)

$$\inf_{x \in X} F_n(x) \rightarrow \min_{x \in X} F_\infty(x) \quad (\text{convergence of minima})$$

(3) Assume that $x_n \rightarrow x_\infty$ and x_n is a "quasi minimum" for $F_n(x)$, namely

$$\lim_{n \rightarrow +\infty} (F_n(x_n) - \inf_{x \in X} F_n(x)) = 0$$

Then $x_\infty \in \arg\min_{x \in X} F_\infty(x)$ (convergence of minimizers)

$$\limsup_{n \rightarrow \infty} F_n(x_n) = \limsup_{n \rightarrow \infty} \left[\sup_{x \in X} F_n(x) + \underbrace{F_n(x_n) - \sup_{x \in X} F_n(x)}_{\downarrow 0} \right]$$

$$= \limsup_{n \rightarrow \infty} \sup_{x \in X} F_n(x)$$

$$(2) = \min_{x \in X} F_\infty(x)$$

On the other hand $\limsup_{n \rightarrow \infty} F_n(x_n) \geq \liminf_{n \rightarrow \infty} F_n(x_n) \geq F_\infty(x_\infty)$

This proves that $F_\infty(x_\infty) \leq \min_{x \in X} F_\infty(x) \Rightarrow x_\infty \in \arg\min.$

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Calculus of Variations —

LECTURE 24

Note Title

04/11/2016

LAGRANGE MULTIPLIERS in $\mathbb{R}^n$

Setting : ball in $\mathbb{R}^n$: $B_r(x_0)$ $x_0 \in \mathbb{R}^n$ $r > 0$
 $f: B_r(x_0) \rightarrow \mathbb{R}$ (closed ball)

$\Phi_1, \dots, \Phi_k: B_r(x_0) \rightarrow \mathbb{R}$ s.t. $\Phi_1(x_0) = \dots = \Phi_k(x_0) = 0$
 Define

$$V := \{x \in B_r(x_0) : \underbrace{\Phi_1(x) = \dots = \Phi_k(x) = 0}_{\text{eqns of } V}\}$$

Theorem Assume that $f, \Phi_1, \dots, \Phi_k$ are of class C^1 in $B_r(x_0)$.
 Assume

$$x_0 \in \operatorname{argmin} \{f(x) : x \in V\}$$

Then

- either $\nabla \Phi_1(x_0), \dots, \nabla \Phi_k(x_0)$ are NOT linearly indep.
- or $\nabla f(x_0) \in \operatorname{span} \{\nabla \Phi_1(x_0), \dots, \nabla \Phi_k(x_0)\}$, namely there exist $\lambda_1, \dots, \lambda_k$ such that

$$\nabla f(x_0) = \lambda_1 \nabla \Phi_1(x_0) + \dots + \lambda_k \nabla \Phi_k(x_0)$$

Proof. Step 1 wlog we can assume that x_0 is a strict local minimum, namely

$$f(x) > f(x_0) \quad \forall x \in V \setminus \{x_0\}$$

If not, replace $f(x)$ by $\hat{f}(x) := f(x) + \|x - x_0\|^2$. Now $\hat{f}$ has a strict minimum at x_0 and
 $\nabla f(x_0) = \nabla \hat{f}(x_0)$.

Step 2 Let us consider the new function

$$F_n(x) := f(x) + n\Phi_1^2(x) + \dots + n\Phi_k^2(x) \quad x \in B_n(x_0)$$

Claim: for every $n \geq 1$, the function F_n has a minimum in $B_n(x_0)$ (trivial: F_n is cont. and $B_n(x_0)$ is compact) and

$$\min \{F_n(x) : x \in B_n(x_0)\} \leq f(x_0) \quad (*)$$

(this is because $\min \leq F_n(x_0) = f(x_0)$)

Step 3 For every $n \in \mathbb{N}$, let x_n denote a min. point of F_n .

Up to subsequences (not relabeled) $x_n \rightarrow x_\infty$

Claim: $x_\infty = x_0$.

Let us prove that $x_\infty \in V$. If not $\exists i \in \{1, \dots, k\}$ s.t. $\Phi_i(x_\infty) \neq 0$ and therefore

$$\liminf_{n \rightarrow \infty} F_n(x_n) \geq \liminf_{n \rightarrow \infty} \underbrace{f(x_n)}_{\downarrow f(x_\infty)} + n \underbrace{\Phi_i^2(x_n)}_{\substack{\uparrow \Phi_i^2(x_\infty) \\ \downarrow +\infty}}$$

and this contradicts $(*)$

Let us prove that actually $x_\infty = x_0$.

$$\liminf F_n(x_n) \geq \liminf f(x_n) = f(x_\infty)$$

and once again this contradicts $(*)$

Step 4 Since $x_n \rightarrow x_0$, then eventually $x_n \in \text{Int}(B_n(x_0))$ and hence

$$\nabla F_n(x_n) = 0 \quad \text{for } n \text{ large}$$

$$\nabla F_m(x_m) = \nabla f(x_m) + 2m \phi_1(x_m) \nabla \phi_1(x_m) + \dots + 2m \phi_k(x_m) \nabla \phi_k(x_m) = 0$$

General fact about linear algebra:

→ or $\nabla f(x_0) \in \text{Span} \{ \dots \}$

$$M_m = (\nabla \varphi(x_m) \mid \nabla \phi_1(x_m) \mid \dots \mid \nabla \phi_k(x_m))$$

Now $M_n \rightarrow M_\infty$ and we used that $\text{rank}(M_\infty) \leq k$.

$\text{rank} \leq k \Leftrightarrow$ all minors $(k+1) \times (k+1)$ are $= 0$
and this is a closed relation.

— 0 — 0 — 0

Lecture 24

Interpretation by means of T -conv.

$$T\text{-}\lim_{n \rightarrow +\infty} F_n(x) = F_\infty(x) := \begin{cases} f(x) & \text{if } x \in V \\ +\infty & \text{otherwise} \end{cases}$$

What we have proved is the convergence of minima and minimizers, because x_0 is the unique element in $\arg\min F_\infty$

Example

$$M_n = \min \left\{ n \int_0^1 u^2 + \int_0^1 \arctan u : u \in C^1, u(0)=0, u(1)=7 \right\}$$

Fact 1 M_n is well defined for every $n \in \mathbb{N}$

(Standard application of direct method)

Fact 2 $\lim_{n \rightarrow +\infty} M_n = +\infty$

$$F_n(u) := \int_0^1 u^2 + \frac{1}{n} \int_0^1 \arctan u$$

$$M_n = n \cdot \min \{ F_n(u) : u(0)=0, u(1)=7 \}$$

Assume that $F_n \xrightarrow{T} F_\infty$ with equicoerc. Then

$$\lim_{n \rightarrow +\infty} \frac{M_n}{n} = \min \{ F_\infty(u) \}$$

In this case the T -lim is also the unif. lim. (for ex in C^0)

$$F_\infty(u) = \int_0^1 u^2 \quad \text{if } u \text{ in } H^1 \text{ satisfies BC}$$

$$\min F_\infty = 49 \quad \Rightarrow \quad M_n = 49n + o(n)$$

Calculus of Variations – LECTURE 25

Note Title

09/11/2016

Four notions of minimum point

Setting: integral functionals

$$F(u) := \int_a^b L(x, u, \dot{u}) dx$$

$$\text{in } X = \{u \in C^1([a, b]) : u(a) = A, u(b) = B\}$$

Def. We say that $u_0 : [a, b] \rightarrow \mathbb{R}$ is

- **GLOBAL MINIMUM** (GM) if $F(u) \geq F(u_0)$ for every $u \in X$,
- **STRONG LOCAL MINIMUM** (SLM) if u_0 is a loc. min. with respect to the C^0 metric, namely

$$\exists \delta > 0 \text{ s.t. } F(u) \geq F(u_0) \quad \forall u \in X \text{ with } \|u - u_0\|_{C^0} \leq \delta$$

$\sup_{x \in [a, b]} |u(x) - u_0(x)|$

- **WEAK LOCAL MINIMUM** (WLM) ... C^1 metric ...

$$\exists \delta > 0 \text{ s.t. } F(u) \geq F(u_0) \quad \forall u \in X \text{ with } \|u - u_0\|_{C^0} \leq \delta$$

$\|\dot{u} - \dot{u}_0\|_{C^0} \leq \delta$

- **DIRECTIONAL LOCAL MINIMUM** (DLM) if the function

$$\psi(t) := F(u_0 + t v) \quad \psi : (-\delta, \delta) \rightarrow \mathbb{R}$$

has a local minimum for $t = 0$.

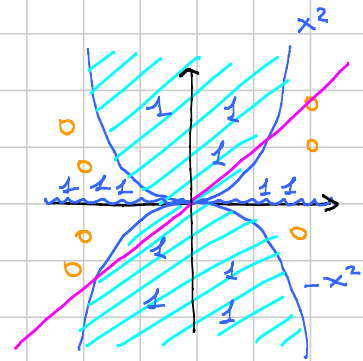
Remark It is easy to see that

$$GM \Rightarrow SLM \Rightarrow WLM \Rightarrow DLM$$

All other implications are false

Example 1 Consider $f: \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(x,y) := \begin{cases} 1 & (x,y) \in \text{shaded region (including } y=0) \\ 0 & \text{otherwise} \end{cases}$$



The origin is a DLM but not a local min. in any sense.

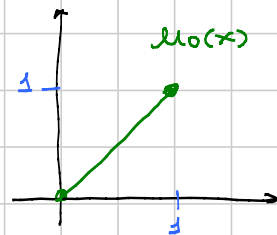
Example 2 $F(u) = \int_0^1 \dot{u}^3 dx$ $u(0)=0, u(1)=1$

We claim that $u_0(x) := x$ is a

SLM but not a WLM.

WLM

SLM (abolished after video)

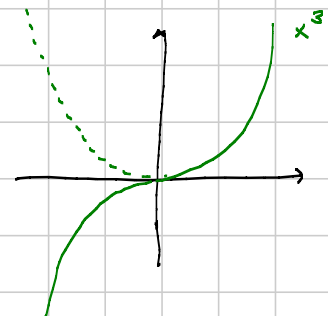


SLM Claim: $u_0(x)$ is the minimum point in
WLM (after video)

$$\mathcal{U} = \{u \in C^1([0,1]) : \dot{u}(x) \geq 0, u(0)=0, u(1)=1\}$$

Proof. Let us consider

$$G(u) := \int_0^1 |\dot{u}|^3 dx$$



$u_0(x)$ is a minimum point for $G(u)$ subject to the BC. It is also the unique GM because the Lagrangian is strictly convex.

On the other hand $F(u) = G(u)$ in $\mathcal{U}$, and therefore

$$F(u) = G(u) \geq G(u_0) = F(u_0) \quad \forall u \in U.$$

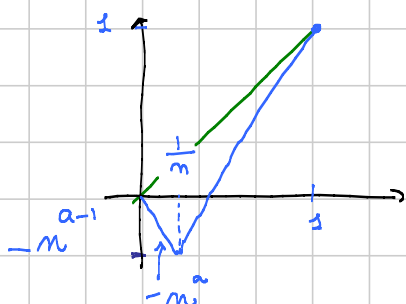
WLM

SLM

We claim that $u_0(x) = x$ is NOT a WLM, namely in any C^0 -neighborhood of u_0 there exists $u(x)$ with $F(u) < F(u_0)$

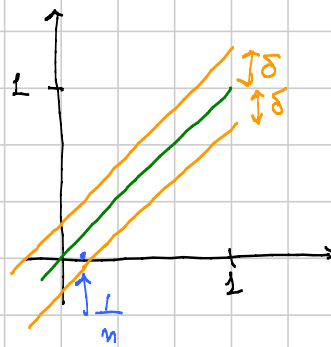
SLM ← after video

Proof. Consider u like below



If $a < 1$, then $u_n(x)$ is as close as we want to $u_0(x)$

$$\begin{aligned} F(u_n) &= \int_0^{1/n} (-n^a)^3 + \int_{1/n}^1 u_n^3 \\ &= -n^{3a-1} + \int_{1/n}^1 u_n^3 \\ &\quad \underbrace{\qquad\qquad\qquad}_{-\infty \text{ if } a > \frac{1}{3}} \quad \underbrace{\qquad\qquad\qquad}_{1 \text{ if } a < 1} \end{aligned}$$



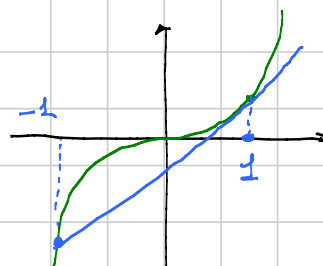
Conclusion: fix any $a \in (\frac{1}{3}, 1)$. Then $u_n \rightarrow u_0$ in the C^0 -norm, but $F(u_n) \rightarrow -\infty$

Remark ① It is not difficult to construct a sequence of C^1 functions converging to u_0 in C^0 .

② The sequence u_n does not belong eventually to any C^1 -ubd of u_0 .

③ We can prove that u_0 is a SLM wrt a C^1 -ubd larger than U . Claim: u_0 is wlm in

$$U_1 := \{u \in C^1 : u(x) \geq -1 + B.c.\}$$

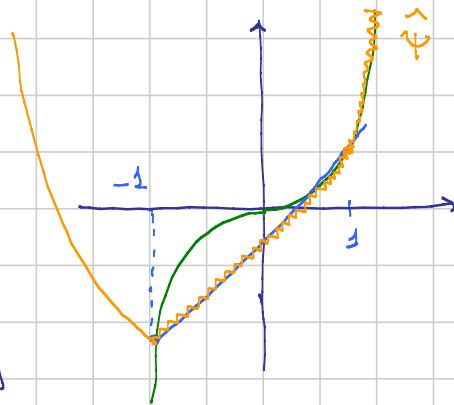


Proof. The same as before with

$$\hat{G}(u) := \int_0^1 \hat{\psi}(u) du$$

where $\hat{\psi}$ is any convex function s.t.

$$\hat{\psi}(x) = \begin{cases} x^3 & x \geq 1 \\ \text{tang. line} & x \in [-1, 1] \\ \dots & x \leq -1 \end{cases}$$



One again $F(u) \geq \hat{G}(u) \geq \hat{G}(u_0) = F(u_0)$

$\uparrow$ $\uparrow$ $\uparrow$
 $\forall u \in G_1$ convexity $\hat{\psi}(1) = 1$
 $\text{---} \circ \text{---} \circ \text{---}$

SECOND VARIATION OF A FUNCTIONAL

$F(u)$ any functional, v be any admissible direction

$$\psi(t) := F(u+tv) \quad t \in (-\delta, \delta)$$

The second variation of F in the point u wrt the dir v is

$$\delta^2 F(u, v) := \psi''(0)$$

If $F(u) = \int_a^b L(x, u, \dot{u}) dx$ with DBC, then

$$F(u+tv) = \int_a^b L(x, \underbrace{u+tv}_s, \underbrace{\dot{u}+t\dot{v}}_p) dx$$

$$\psi'(t) = \int_a^b L_s(x, u+tv, \dot{u}+t\dot{v}) v + L_p(x, u+tv, \dot{u}+t\dot{v}) \dot{v}$$

$$\psi''(t) = \int_a^b L_{ss}(x, u+tv, \dot{u}+t\dot{v}) v^2 + 2 L_{sp}(\dots) v \dot{v} + L_{pp}(\dots) \dot{v}^2$$

In conclusion, for the standard functional $L(x, s, p)$

$$\delta^2 F(u, v) = \int_a^b L_{ss}(x, u, \dot{u}) v^2 + 2 L_{sp}(\dots) v \dot{v} + L_{pp}(\dots) \dot{v}^2$$

Prop. Assume that $L(x, s, p)$ is of class C^2 .

Assume that u_0 is a DLM for $F(u)$.

Then

- $u_0(x)$ is a solution to the Euler-Lagrange eqn

$$\psi'(0) = \delta F(u_0, v) = 0 \quad \forall v \text{ admissible}$$

- $u_0(x)$ satisfies

$$\psi''(0) = \delta^2 F(u_0, v) \geq 0 \quad \forall v \text{ admissible}$$

Remark When u_0 is fixed, the second variation is a quadratic functional in v .

Calculus of Variations

LECTURE 26

Note Title

09/11/2016

QUADRATIC FUNCTIONALS

Def. A quadr. functional is the following

$$Q(v) := \int_a^b A(x) v^2 + 2B(x) v \dot{v} + C(x) \dot{v}^2$$

where $A(x)$, $B(x)$ and $C(x)$ are given functions defined in $[a, b]$.

Remark We consider different assumptions on A, B, C

- A, B, C in $L^\infty([a, b])$ is enough in order to define $Q(v)$
- A, B, C in C^0
- A, B, C in C^1

Def. A quadratic functional is nonnegative (positive semi-def.) if

$$Q(v) \geq 0 \quad \forall v \in C_c^\infty([a, b])$$

Prop. The following are equivalent

- (i) $Q(v) \geq 0$ for every $v \in C_c^\infty([a, b])$,
- (ii) $Q(v) \geq 0$ " " $v \in C^1([a, b])$ with $v(a) = v(b) = 0$,
- (iii) $Q(v) \geq 0$ " " $v \in PNC^1([a, b])$ " "
- (iv) $Q(v) \geq 0$ " " $v \in H_0^1([a, b])$ " "

Proof Standard approximation argument.

Goal of the theory: discovering necessary/suff. conditions on $A(x)$, $B(x)$, $C(x)$ that guarantee that $Q(v)$ is nonnegative.

LEGENDRE CONDITION

$$(L) \quad C(x) \geq 0 \quad \forall x \in [a, b]$$

$$(L^+) \quad C(x) > 0 \quad \forall x \in [a, b]$$

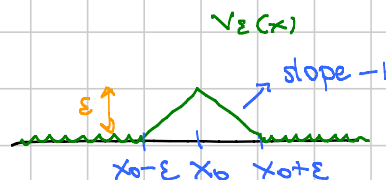
Theorem 1 Assume that A, B, C are continuous. Then

$$Q(v) \text{ nonnegative} \Rightarrow (L)$$

Proof. We know that $Q(v) \geq 0$ for every $v \in PWC^1$.
Take any $x_0 \in (a, b)$ and for $\varepsilon > 0$
define v_ε as in the figure

$$0 \leq Q(v_\varepsilon)$$

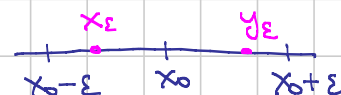
$$\begin{aligned} &= \int_{x_0-\varepsilon}^{x_0} A(x) v_\varepsilon^2 + 2B(x) v_\varepsilon(x) + C(x) \\ &+ \int_{x_0}^{x_0+\varepsilon} A(x) v_\varepsilon^2 - 2B(x) v_\varepsilon(x) + C(x) \end{aligned}$$



Note that v_ε is admissible if ε is small enough.

Divide by ε and apply the mean value theorem for integrals

$$\begin{aligned} 0 \leq & A(x_\varepsilon) v_\varepsilon^2(x_\varepsilon) + 2B(x_\varepsilon) v_\varepsilon(x_\varepsilon) + C(x_\varepsilon) \\ & + A(y_\varepsilon) v_\varepsilon^2(y_\varepsilon) - 2B(y_\varepsilon) v_\varepsilon(y_\varepsilon) + C(y_\varepsilon) \end{aligned}$$



Therefore $C(x_0) \geq 0$.

Example $Q(v) = \int_0^l \sin x \cdot v^2 + \cos x \cdot \dot{v}^2$

This cannot be nonnegative if $l \geq \frac{\pi}{2}$

JACOBI CONDITION and CONJUGATE POINTS

Let us assume that A, B, C are of class C^1 . Consider the ODE

$$(C\dot{v} + Bv)' = B\dot{v} + Av \quad (\text{JDE})$$

This is called JACOBI DIFF. EQN

Note: this is exactly Euler-Lagrange eqn for $\mathcal{Q}(v)$

$$L(x, s, p) : A(x) s^2 + 2B(x) sp + C(x) p^2$$

Remark If A, B, C and v are regular enough we can write JDE as

$$C'\dot{v} + C\ddot{v} + B'\dot{v} + \cancel{B\dot{v}} = \cancel{B\dot{v}} + Av$$

Under assumption (L^+) we can write it in normal form

$$\ddot{v} = \frac{C'}{C} \dot{v} + \frac{A - B'}{C} v$$

This is a second order linear eqn of the form

$$\ddot{v} = \alpha(x)\dot{v} + \beta(x)v$$

for which the usual theory (existence, uniqueness, gl. ex.) applies.

Def. We say that a point c is conjugate to a if (JDE) has a solution such that

$$v(a) = 0, \quad \dot{v}(a) = 1, \quad v(c) = 0$$

In other words, solve (JDE) with initial data $v(a)=0, \dot{v}(a)=1$.
All other 0's of v are conjugate points of a .

Remark Solving with $v(a)=0, \dot{v}(a)=k \neq 0$ produces a multiple of the previous solution, and therefore the same conj. points.

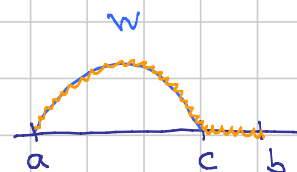
JACOBI CONDITIONS (J) No conj. points of a in (a,b) .
(J⁺) " " " " " " $(a,b]$

Theorem Assume A, B, C of class C^1 .
Assume that (L^+) is satisfied
Assume that $Q(v)$ is nonnegative.
Then (J) is satisfied.

Proof. Assume by contradict. that a conj. point exists in (a,b) .
Let w be the con. solution of (JDE) with
 $w(a)=0 \quad \dot{w}(a)=1 \quad w(c)=0$

Define

$$v(x) := \begin{cases} w(x) & \text{if } x \in [a, c] \\ 0 & \text{if } x \in [c, b] \end{cases}$$



Claim: $Q(v) = 0$. Suppose this is true.

Then v is a GM for Q , and hence it is a solution to Euler-Lagrange eqn of Q which is JDE. But $v(x)$ is not of class C^2 in $x=c$, but all solutions to JDE are of class C^2 in $[a,b]$.

(note that $\dot{w}(c) \neq 0$ because otherwise $w(x) \equiv 0$).

In order to prove the claim it is enough to prove that

$$\int_a^c A w^2 + 2B w \dot{w} + C \dot{w}^2 = 0$$

This follows from JDE after integration by parts

$$(C\dot{w} + Bw)' \cdot w = (B\dot{w} + Aw)w$$

$$\int_a^c (C\dot{w} + Bw)' w = \int_a^c Bw\dot{w} + Aw^2$$

$$-\int_a^c C\dot{w}^2 + Bw\dot{w} + \text{boundary term which is 0 because } w=0 \text{ in } a \text{ and } c.$$

Remark We could define conj. points relative to b by solving JDE with "initial" data

$$w(b) = 0 \quad \dot{w}(b) = \pm 1$$

Fact. There exists a point $c \in (a, b)$ conj. to $a \iff$
 " " " " " " to b

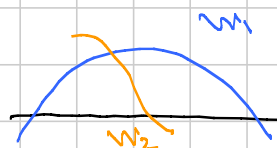
(under assumption (L^+))

Lemma (oscillation lemma) Take any two lin. indep. solutions w_1 and w_2 to a lin. equ.

$$\ddot{v} = \alpha(x)\dot{v} + \beta(x)v$$

Then between any two zeroes of w_1 there exists at least one zero of w_2

— o — o —



Calculus of Variations – LECTURE 27

Note Title

11/11/2016

$$Q(v) = \int_a^b A(x) v^2 + 2B(x) v\dot{v} + C(x) \dot{v}^2$$

$$(L) \quad C(x) \geq 0 \quad \forall x \in [a, b]$$

$$(L^+) \quad C(x) > 0 \quad \forall x \in [a, b]$$

$$(JDE) \quad [C(x)\dot{v} + B(x)v]' = A(x)v + B(x)\dot{v}$$

$\leadsto$ Conjugate points (wrt a or b)

(J) No conj. point in (a, b)

(J⁺) No conj. point of a in $[a, b]$

Necessary conditions

$$Q(v) \geq 0 \quad \forall v \in C_c^\infty([a, b]) \Rightarrow (L) + [(L^+) \Rightarrow (J)]$$

$(L^+) \Rightarrow$ can write (JDE) in normal form $\Rightarrow$ ex./uniq./reg.

Sufficient conditions

$$(L^+) + (J^+) \Rightarrow Q(v) \geq 0 \quad \forall v \in C_c^\infty([a, b])$$

ROAD MAP

$$\textcircled{1} \quad (L^+) + (J^+) \Rightarrow (J^{++})$$

$$\textcircled{2} \quad (J^{++}) \Rightarrow (R)$$

$$\textcircled{3} \quad (R) \Rightarrow Q(v) \text{ is nonnegative}$$

(J⁺⁺) There exists a solution w to (JDE) such that
 $w(x) > 0 \quad \forall x \in [a, b]$

(R) There exists $z: [a, b] \rightarrow \mathbb{R}$ solution to

$$z' = \frac{[z + B(x)]^2}{C(x)} - A(x) \quad (\text{RDE})$$

↑
RICCATI

Remark 1 - (L⁺) allows to divide by $C(x)$

2 - The RHS of (RDE) is quadratic in z : no guarantee that solutions starting from a survive up to b .

Proof of ① Due to (J⁺) we know that (JDE) has a solution $w_1: [a, b] \rightarrow \mathbb{R}$ s.t.

$$w_1(x) > 0 \quad \forall x \in [a, b]$$

In a similar way (oscillation lemma) we know that (JDE) has a solution w_2 with

$$w_2(x) > 0 \quad \forall x \in [a, b]$$

Since (JDE) is linear, $w_1 + w_2$ is again a solution and it is strictly positive in $[a, b]$.

Proof of ② Let us take w as in step ①. Let us set

$$z(x) := - \frac{[C(x)\dot{w}(x) + B(x)w(x)]}{w(x)} \quad \leftarrow \text{LHS of (JDE)}$$

$$= - C(x) \frac{\dot{w}(x)}{w(x)} - B(x)$$

Claim: z solves (RDE).

$$\text{RHS of (RDE)}: \frac{[z + B]^2}{C} - A = C \frac{\dot{w}^2}{w^2} - A$$

$$\text{LHS of (RDE)}: z' = - \frac{[C\dot{w} + Bw]'w - \dot{w}[C\dot{w} + Bw]}{w^2}$$

$$\text{use (JDE)} \leftarrow = - \frac{[B\dot{w} + A w]w - \dot{w}[C\dot{w} + Bw]}{w^2} = -A + C \frac{\dot{w}^2}{w^2}$$

VERY SUFFICIENT CONDITION Assume $C(x) \geq 0$ in $[a, b]$ and assume that

$$A(x) \cdot C(x) \geq B(x)^2 \quad \forall x \in [a, b]$$

Then $Q(v) \geq 0$ for every $v \in H^1([a, b])$ indep. of BC .

Proof. The condition guarantees that the matrix

$$\begin{pmatrix} A(x) & B(x) \\ B(x) & C(x) \end{pmatrix}$$

is a nonnegative matrix (i.e. nonnegative eigenvalues) and therefore

$$A(x) v^2(x) + 2 B(x) v(x) \dot{v}(x) + C(x) \dot{v}^2(x) \geq 0$$

for every $x \in [a, b]$.

— o — o —

Proof of ③ Let $z(x)$ be a solution to (RDE) defined in $[a, b]$

Let us consider

$$\begin{aligned} \hat{Q}(v) &:= Q(v) + \int_a^b [z(x) v^2(x)]' \\ &= \int_a^b [A(x) + \ddot{z}(x)] v^2 + 2 [B(x) + z(x)] v \dot{v} + C(x) \dot{v}^2 \end{aligned}$$

We observe that $\hat{Q}(v) = Q(v)$ for every admissible v .

This is because

$$\int_a^b [z(x) v^2(x)]' dx = [z(x) v^2(x)]_a^b = 0 \quad \text{because of } BC \text{ on } v.$$

Surprise: $\tilde{Q}$ satisfies the "very sufficient condition":

$$[A(x) + \tilde{z}(x)] \cdot C(x) \geq [B(x) + \tilde{z}(x)]^2$$

Asking = is just (RDE)

— 0 — 0 —

Comment 1 (JDE) vs (RDE)

Riccati equ $z' = A(x)z^2 + B(x)z + C(x)$

Second order linear ODE : $v'' = \alpha(x)v' + \beta(x)v$

Assume we want a solution to 2nd order ODE which is nonnegative. Reasonable: $v = e^w$. Let us write the equ. for w :

$$\dot{v} = \dot{w}e^w, \quad \ddot{v} = \ddot{w}e^w + \dot{w}^2e^w \quad \text{so that}$$

$$\cancel{\dot{w}e^w} + \dot{w}^2 \cancel{e^w} = \alpha(x) \cancel{\dot{w}e^w} + \beta(x) \cancel{e^w}$$

$$z = \dot{w} \text{ solves } z' + z^2 = \alpha(x)z + \beta(x)$$

which is a Riccati equ. Moreover $v = e^w$ implies $w = \log v$ and therefore

$$z = \dot{w} = \frac{\dot{v}}{v} \quad \rightarrow \text{this reminds us of the substitution we did before}$$

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Oscillation Lemma Assume w_1 and w_2 are two lin. indep. solutions to

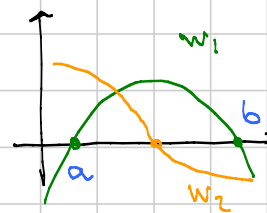
$$\ddot{v} = \alpha(x)\dot{v} + \beta(x)v.$$

Then between any two consecutive zeroes of $w_1(x)$ there is exactly one zero of w_2

Proof. Let us consider

$$W(x) := \det \begin{pmatrix} w_1(x) & \dot{w}_1(x) \\ w_2(x) & \dot{w}_2(x) \end{pmatrix}$$

$$= w_1 \dot{w}_2 - \dot{w}_1 w_2$$



$$\begin{aligned} \dot{W} &= \dot{w}_1 \dot{w}_2 + w_1 \ddot{w}_2 - \ddot{w}_1 w_2 - \dot{w}_1 \dot{w}_2 \\ &= w_1 (\alpha(x) \dot{w}_2 + \beta(x) \ddot{w}_2) - w_2 (\alpha(x) \dot{w}_1 + \beta(x) \ddot{w}_1) \\ &= \alpha(x) W \end{aligned}$$

So W solves $\dot{W} = \alpha(x) W \Rightarrow$ the sign of $W(x)$ is constant

$$W(a) = -\dot{w}_1(a) w_2(a) < 0 \Rightarrow W(b) < 0$$

$$W(b) = -\dot{w}_1(b) w_2(b) < 0 \Rightarrow w_2(b) < 0$$

$\Rightarrow w_2$ changes its sign in $[a, b]$. Why only once?

If not, we swap w_1 and w_2 and we obtain one more 0 of w_1 .

Calculus of Variations

LECTURE 28

Note Title

11/11/2016

Prop. Let us consider a quadratic functional.

Then there exists a constant $M \in \mathbb{R}$ such that

$$|Q(v)| \leq M \int_a^b \dot{v}^2 dx \quad \forall v \text{ admissible } (v(a)=v(b)=0)$$

The constant M depends on $\|A\|_\infty, \|B\|_\infty, \|C\|_\infty$ and $b-a$.

Proof $|Q(v)| \leq \int_a^b |A| v^2 + 2 \int_a^b |B| |v \dot{v}| + \underbrace{\int_a^b |C| \dot{v}^2}_{\leq \|C\|_\infty \int_a^b \dot{v}^2}$

Since $v(a)=v(b)=0$ we can always estimate

$\|v(x)\|_{L^\infty}$ in terms of $\left(\int_a^b \dot{v}^2\right)^{1/2}$... conclusion is standard.

Remark For every $\delta > 0 \exists \varepsilon > 0$ such that

$$\|A\|_\infty \leq \varepsilon, \|B\|_\infty \leq \varepsilon, \|C\|_\infty \leq \varepsilon \Rightarrow M \leq \delta$$

Prop 2 Assume that (L^+) and (J^+) are satisfied.

Then there exists $\varepsilon > 0$ such that

$$Q(v) \geq \varepsilon \int_a^b \dot{v}^2 dx \quad \forall \text{ admissible } v$$

Proof $\tilde{Q}(v) := \int_a^b A v^2 + 2B v \dot{v} + (C - \varepsilon) \dot{v}^2$

Claim: if ε is small enough, then $\tilde{Q}$ still satisfies (L^+) and (J^+)

$$Q \text{ satisfies } (L^+) \Rightarrow \inf_{x \in [a,b]} C(x) > 0 \Rightarrow \inf_{x \in [a,b]} (C(x) - \varepsilon) > 0 \text{ when } \varepsilon \text{ is small}$$

In order to prove (J⁺) we have to solve (JDE) for $\hat{Q}$.
This equation is of the form

$$\ddot{v} = \alpha_\varepsilon(x) \dot{v} + \beta_\varepsilon(x) v \quad (\text{JDE}_\varepsilon)$$

where $\alpha_\varepsilon(x) \rightarrow \alpha(x)$ and $\beta_\varepsilon(x) \rightarrow \beta(x)$ UNIFORMLY as $\varepsilon \rightarrow 0$.

We know that the limit eqn

$$\begin{aligned} \ddot{v} &= \alpha(x) \dot{v} + \beta(x) v \\ v(a) &= 0 \\ \dot{v}(a) &= 1 \end{aligned}$$

has a solution $v(x)$ which does not vanish in $(a, b]$.

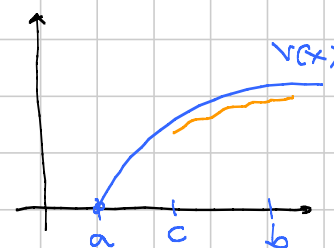
Claim: (JDE_ε) with initial data $v_\varepsilon(a) = 0$, $\dot{v}_\varepsilon(a) = 1$
has the solution which does not vanish in $(a, b]$.

From the theory of ODEs we know that $v_\varepsilon \rightarrow v$ and $\dot{v}_\varepsilon \rightarrow \dot{v}$ unif. in $[a, b]$.

There exists $c \in (a, b)$ s.t.

$$\dot{v}(x) \geq \frac{1}{2} \quad \forall x \in [a, c]$$

$$v(x) \geq m > 0 \quad \forall x \in [c, b]$$



Due to unif. conv., $v_\varepsilon(x) \geq \frac{m}{2} \quad \forall x \in [c, b]$ for ε small

and $\dot{v}_\varepsilon(x) \geq \frac{1}{4} \quad \forall x \in [a, c]$ "

This implies that v_ε is strictly increasing in $[a, c]$ for ε small, and hence it cannot vanish in $(a, c]$.

— o — o —

Prop 3 Assume that $Q(v)$ satisfies (L^+) and (J^+)
 Let us consider a new quadratic functional $\hat{Q}(v)$
 with coeff. $\hat{A}, \hat{B}, \hat{C}$ s.t.

$$\|\hat{A} - A\|_\infty \leq \varepsilon, \quad \|\hat{B} - B\|_\infty \leq \varepsilon, \quad \|\hat{C} - C\|_\infty \leq \varepsilon$$

If $\varepsilon > 0$ is small enough, then $\hat{Q}(v) \geq 0$ for every admissible v .

Proof. Let $\hat{\hat{Q}}(v) := Q(v) - \hat{Q}(v)$, so that

$$\hat{Q}(v) = Q(v) - \hat{\hat{Q}}(v) \geq Q(v) - |\hat{\hat{Q}}(v)|$$

Use that $Q(v) \geq \varepsilon_0 \int_a^b \dot{v}^2$ (Prop. 2)

$$|\hat{\hat{Q}}(v)| \leq M \int_a^b \dot{v}^2 \quad (\text{Prop. 4})$$

If ε is small enough, then M is as small as we need,
 and in particular $M \leq \varepsilon_0$.

This concludes the proof.

Back to second variation Let

$$F(u) := \int_a^b L(x, u(x), \dot{u}(x))$$

$$\delta F(u, v) := \int_a^b \underbrace{L_{ss}(x, u, \dot{u})}_{A(x)} v^2 + 2 \underbrace{L_{sp}(\cdot)}_{B(x)} v \dot{v} + \underbrace{L_{pp}(\cdot)}_{C(x)} \dot{v}^2$$

Necessary conditions

$$\text{no DLM} \Rightarrow (E) + (L) + [(L^+) \Rightarrow (J)]$$

with DBC $\uparrow$
Euler-Lagr
equ

Sufficient conditions

$$(E) + (L^+) + (J^+) \Rightarrow \text{no WLM}$$

Proof. We need to prove that

$$F(u_0 + v) \geq F(u_0)$$

provided that the C^1 -norm of v is small. Let us introduce

$$\psi(t) := F(u_0 + tv)$$

$$\text{Then } F(u_0 + v) = \psi(1) = \psi(0) + \overset{0(E)}{\psi'(0)} \cdot 1 + \frac{1}{2} \psi''(t_0)$$

$\uparrow$
Taylor

$$\text{For some } t_0 \in (0,1) \quad = F(u_0) + \frac{1}{2} \psi''(t_0)$$

So it is enough to prove that $\psi''(t_0) \geq 0$. But

$$\begin{aligned} \psi''(t_0) &= \int_a^b L_{ss}(x, u_0(x) + t_0 v(x), \dot{u}_0(x) + t_0 \dot{v}(x)) v^2(x) + \text{other 2 terms} \\ &= Q^{(t_0)}(v) \end{aligned}$$

Idea: we know that $Q^{(0)}(v)$ satisfies (J^+) and (L^+) , and therefore it is strictly positive.

If L is of class C^3 (but maybe C^2 is enough) and $\|v\|_\infty$ and $\|\dot{v}\|_\infty$ are small, then the coeff. of

$Q^{(t_0)}$ are as close as needed to the coeff. of $Q^{(0)}$.

We conclude by exploiting prop. 3.

— 0 — 0 —

THE Example Consider

$$F(u) = \int_0^l \dot{u}^2 - u^2$$

(L^+) is trivial $C(x) \equiv 1$. (JDE) $\ddot{u} = -u$

Need to solve with $u(0) = 0$, $\dot{u}(0) = 1$

The solution is $u(x) = \sin x$. The first conj. point is π .

Conclusion: 1- if $l > \pi$, then (J) not satisfied, hence $F(u)$ is NOT nonnegative

2- if $l < \pi$, then (L^+) and (J^+) are satisfied, and therefore $F(u) \geq 0$

3- if $l = \pi$, then $F(u) \geq 0$ again.

Proof. If F is NOT nonnegative, then the supremum is $-\infty$. Assume that F_π has $\sup = -\infty$.

This means that

$$F_\pi(u) \leq -2016$$

for some u vanishing at the boundary

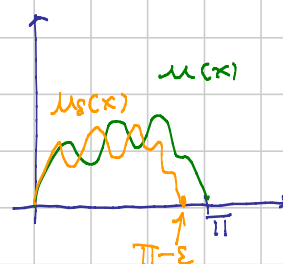
Rescale u to the interval $[0, \pi - \varepsilon]$ with an affine transformation

$$\text{Then } \int_0^{\pi-\varepsilon} \dot{u}_\varepsilon^2 - u_\varepsilon^2 \rightarrow -2016$$

and therefore $F_{\pi-\varepsilon}(u_\varepsilon) < 0$ for ε small.

This contradicts that

$F_{\pi-\varepsilon}$ is nonnegative



Calculus of Variations - LECTURE 29

Note Title

16/11/2016

"One idiot in a crowd probably remains unnoticed, but a bunch of idiots stands out very well"
 [loosely based on De Giorgi]

CALIBRATION

Motivating example 1 $\min \left\{ \int_0^1 u^2 dx : u(0)=0, u(1)=3 \right\}$

The solution is $u_0(x) = 3x$. New proof: consider

$$p^2 \geq 6p - 9 \quad \forall p \in \mathbb{R}$$

Then

$$F(u) = \int_0^1 u^2 dx \geq \int_0^1 (6u - 9) dx = 6[u(x)]_0^1 - 9 = 18 - 9 = 9 = F(u_0)$$

Equality holds if and only if there is equality in $u^2 \geq 6u - 9$ namely iff $u \equiv 3$.

Motivating example 2 Same problem, different ineq.

$$p^2 \geq 2 \frac{3m+5}{m+x} p - \left(\frac{3m+5}{m+x} \right)^2 \quad \forall p \in \mathbb{R}, \forall x \geq 0, \forall m \in \mathbb{R}, \forall m > 0$$

As before

$$\begin{aligned} F(u) = \int_0^1 u^2 dx &\geq \int_0^1 \underbrace{2 \frac{3m+u(x)}{m+x} u - \left(\frac{3m+u(x)}{m+x} \right)^2}_{\frac{d}{dx} \frac{(3m+u(x))^2}{m+x}} dx \\ &= \left[\frac{(3m+u(x))^2}{m+x} \right]_0^1 = \frac{(3m+3)^2}{m+1} - \frac{9m^2}{m} \\ &= 9(m+1) - 9m = 9 = F(u_0) \end{aligned}$$

Remark Example 1 is the limit of example 2 as $m \rightarrow \infty$

Motivating example 3 The same as 2, with $m = 0$

$$p^2 \geq 2 \frac{S}{x} p - \frac{S^2}{x^2}$$

$$\begin{aligned} F(u) = \int_0^1 \dot{u}^2 dx &\geq \int_0^1 2 \frac{u(x)}{x} \dot{u}(x) - \frac{u(x)^2}{x^2} \\ &= \left[\frac{u(x)^2}{x} \right]_0^1 = 9 = F(u_0) \end{aligned}$$

$\uparrow$
 because
 $u(x)^2 = o(x)$ as $x \rightarrow 0$

— 0 — 0 —

NULL LAGRANGIAN Consider a lagrangian of the form

$$\hat{L}(x, s, p) = A(x, s)p + B(x, s)$$

Assume now that $A_x(x, s) = B_s(x, s) \quad \forall (x, s) \in [a, b] \times \mathbb{R}$

Then there exists a function $V(x, s)$ s.t.

$$A(x, s) = V_s(x, s)$$

$$B(x, s) = V_x(x, s)$$

and therefore

$$\begin{aligned} \int_a^b \hat{L}(x, u(x), \dot{u}(x)) dx &= \int_a^b V_s(x, u(x)) \dot{u}(x) + V_x(x, u(x)) dx \\ &= \left[V(x, u(x)) \right]_a^b \\ &= V(a, A) - V(b, B) \end{aligned}$$

Therefore, the integral functional depends only on DBC of u .

Lemma (Calibration through null Lagrangian)

Let $L(x, s, p)$ be any Lagrangian.

Assume there exist $A(x, s)$ and $B(x, s)$ s.t.

(i) $A_x = B_s$ in $[a, b] \times \mathbb{R}$,

(ii) $L(x, s, p) \geq A(x, s)p + B(x, s)$ in $[a, b] \times \mathbb{R}^2$,

(iii) $L(x, u_0(x), \dot{u}_0(x)) = A(x, u_0(x))\dot{u}_0(x) + B(x, u_0(x))$ in $[a, b]$.

Then $u_0(x)$ is a minimizer of

$$F(u) = \int_a^b L(x, u(x), \dot{u}(x)) dx$$

subject to its DBC.

Proof. As in the examples for every u

$$\begin{aligned} F(u) &= \int_a^b L(x, u(x), \dot{u}(x)) dx && \text{(use (ii))} \\ &\geq \int_a^b [A(x, u(x))\dot{u}(x) + B(x, u(x))] dx && \text{(null Lagrangian)} \\ &= \left[V(x, u(x)) \right]_a^b && \text{if } u \text{ and } u_0 \text{ have same DBC} \\ &= \left[V(x, u_0(x)) \right]_a^b && \text{(null Lagrangian)} \\ &= \int_a^b [A(x, u_0(x))\dot{u}_0(x) + B(x, u_0(x))] dx && \text{(use (iii))} \\ &= \int_a^b L(x, u_0(x), \dot{u}_0(x)) dx \\ &= F(u_0). \end{aligned}$$

— o — o —

Remark If v_0 is any other minimizers with the same DBC, then v_0 has to satisfy the equality in (iii).

Previous example 1 Quadratic functional

$$L(x, s, p) = A(x)s^2 + 2B(x)sp + C(x)p^2$$

$$V(x, s) = -z(x)s^2 \rightsquigarrow \hat{L}(x, s, p) = V_s(x, s)p + V_x(x, s) \\ = \underbrace{-2z(x)}_{A(x, s)}sp - \underbrace{z(x)}_{B(x, s)}s^2$$

Condition (i) is trivial

Condition (iii) $L(x, u_0(x), \dot{u}_0(x)) = \hat{L}(x, u_0(x), \dot{u}_0(x))$ is trivial when $u_0(x) \equiv 0$

Condition (ii) $L(x, s, p) \geq \hat{L}(x, s, p)$ is ok if $z(x)$ is the solution to the Riccati eqn in the proof of the suff. condition.

Previous example 2 Assume that $L(x, s, p)$ is convex in (s, p) for every $x \in [a, b]$. Then

$$L(x, s, p) \geq L(x, u_0(x), \dot{u}_0(x)) + (s - u_0(x)) L_s(x, u_0(x), \dot{u}_0(x)) \\ + (p - \dot{u}_0(x)) L_p(\dots) \\ = \hat{L}(x, s, p)$$

Claim: $\hat{L}(x, s, p)$ is a null Lagrangian

$$\hat{L}(x, s, p) = \underbrace{L_p(x, u_0(x), \dot{u}_0(x))}_{A(x, s)}p + \underbrace{L(x, u_0(x), \dot{u}_0(x)) + (s - u_0(x)) L_s(\dots) - \dot{u}_0(x) L_p(\dots)}_{B(x, s)}$$

Let us check (i), (ii), (iii).

- (iii) is trivial
- (ii) is the inequality coming from convexity
- (i) $A_x = B_s \Leftrightarrow \frac{d}{dx} L_p(x, u_0(x), \dot{u}_0(x)) = L_s(\dots) \Leftrightarrow u_0$ satisfies (E)

Back to motivating example 1

$$p^2 \geq 6p - 9$$

$$V(x, s) := 6s - 9x$$

$$\int_a^b \dot{u}(x)^2 dx = V(b, u(b)) - V(a, u(a))$$

Let us consider the level sets of V : $6s - 9x = c$, $s = \frac{3}{2}x + c$

Any other solution with

$(a, u(a)) \in \text{Line 1}$

$(b, u(b)) \in \text{Line 2}$

satisfies the same equality,
and therefore $u_0(x) = 3x$
minimizes

$$F(u) = \int_a^b \dot{u}^2$$

among all functions u with endpoints in Line 1 and Line 2.

In this way we actually have proved minimality for a curve to curve problem.

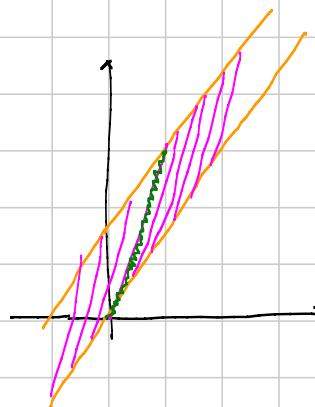
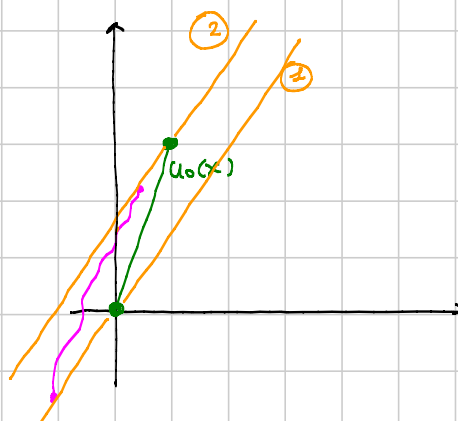
Moreover, we have equality iff $\dot{u} \equiv 3$, and therefore all functions

$$u(x) = 3x + k$$

are minimum points as well.

Notation $V(x, s)$ is usually
called "verification function"

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Calculus of Variations

LECTURE 30

Note Title

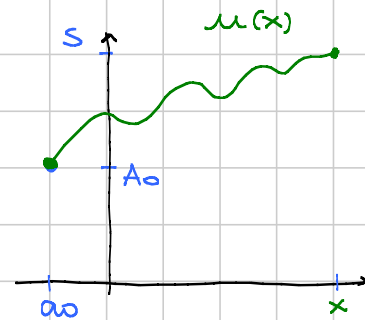
16/11/2016

VALUE FUNCTION Assume $L(x, s, p)$ defined in $\mathbb{R}^3$.

let us fix a "base point" (a_0, A_0)

For every $(x, s) \in \mathbb{R}^2$ let us solve

$$\min \left\{ \int_{a_0}^x L(t, u(t), \dot{u}(t)) dt : u(a_0) = A_0, u(x) = s \right\}$$



let $V(x, s)$ denote the minimum value.

↑ value function

Theorem Assume that $V(x, s)$ is of class C^1 .

Then it is a verification function, namely the null Lagrangian

$$\hat{L}(x, s, p) := V_s(x, s)p + V_x(x, s)$$

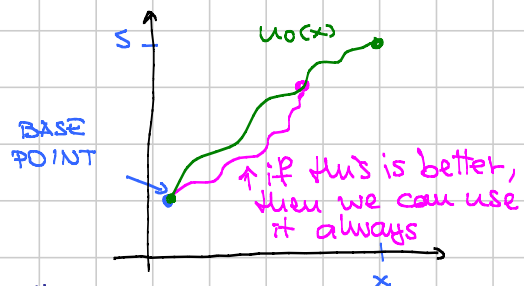
satisfies

$$(ii) \quad L(x, s, p) \geq \hat{L}(x, s, p)$$

$$(iii) \quad L(x, u_0(x), \dot{u}_0(x)) = \hat{L}(x, u_0(x), \dot{u}_0(x)) \quad \text{for every } u_0(x) \text{ which realizes the minimum.}$$

Proof. (iii) Assume that

$$\int_{a_0}^x L(t, u_0(t), \dot{u}_0(t)) dt = V(x, s)$$



Then the same relation is true for all points on the graph of u_0 . This proves that

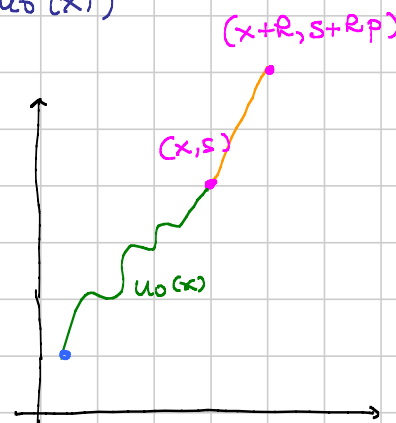
$$\int_{a_0}^x L(t, u_0(t), \dot{u}_0(t)) dt = V(x, u_0(x))$$

If we compute the derivative wrt x we find

$$\begin{aligned} L(x, u_0(x), \dot{u}_0(x)) &= V_x(x, u_0(x)) + V_s(x, u_0(x)) \dot{u}_0(x) \\ &= \hat{L}(x, u_0(x), \dot{u}_0(x)) \end{aligned}$$

which proves (iii).

$$\begin{aligned} V(x+R, s+Rp) &\leq V(x, s) + \\ &\quad + \int_0^R L(x+t, s+tp, p) dt \end{aligned}$$



Dividing by R we obtain

$$\frac{V(x+R, s+Rp) - V(x, s)}{R} \leq \frac{1}{R} \int_0^R L(x+t, s+tp, p) dt$$

When $R \rightarrow 0$ we find that

$$\frac{V_x(x, s) + V_s(x, s)p}{\hat{L}(x, s, p)} \leq L(x, s, p)$$

This proves (ii).

— o — o —

In practice: suppose we need to minimize

$$F(u) = \int_a^b L(x, u, \dot{u}) dx$$

subject to some DBC. Suppose we suspect that some $u_0(x)$ is the minimum. Suppose we think we can solve the problem for any choice of DBC. Then we can compute the value function, and from the value function we can obtain a calibration $\hat{L}(x, s, p)$. If we can check (ii) and (iii), then minimality is proved.

Example $\min \left\{ \int_0^1 \dot{u}^2 : u(0)=0, u(1)=3 \right\}$

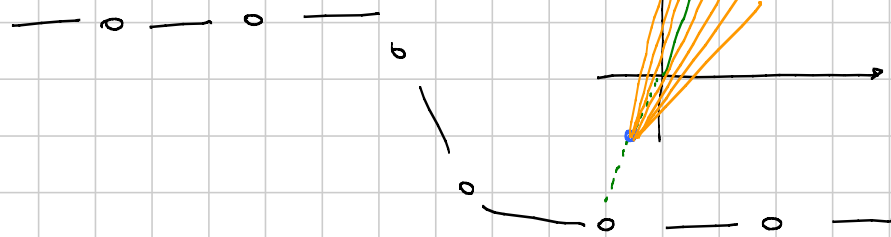
Fix a base point and compute $V(x,s)$ using lines

$$V(x,s) = \left[\frac{(s+3u)}{(x+u)} \right]^2 (x+u)$$

+
Derivative

$$= \frac{(s+3u)^2}{(x+u)}$$

as in example 2 of previous lecture



Develop necessary / sufficient conditions for $u_0(x)$ to be SLM

Def. (WEIERSTRASS EXCESS FUNCTION)

Let $L(x,s,p)$ be the usual Lagrangian

The WEF is defined as

$$E(x,s,p,q) := L(x,s,q) - L(x,s,p) - (q-p)L_p(x,s,p)$$

$$x \in [a,b], (p,q,s) \in \mathbb{R}^3$$

Remark Looks like Taylor expansion of L wrt last variable.

In particular, if $p \rightarrow L(x,s,p)$ is convex for some (x,s) then

$$E(x,s,p,q) \geq 0 \quad \forall (p,q) \in \mathbb{R}^2$$

WEIERSTRASS CONDITIONS

$$(W) \quad E(x, u_0(x), \dot{u}_0(x), q) \geq 0 \quad \forall (x, q) \in [a, b] \times \mathbb{R}$$

$$(W^+) \quad \exists \delta > 0 \text{ s.t. } E(x, s, p, q) \geq 0 \\ \text{for every } (x, s, p, q) \in [a, b] \times \mathbb{R}^3 \text{ s.t. } |s - u_0(x)| \leq \delta \\ |p - \dot{u}_0(x)| \leq \delta$$

Necessary condition

$$u_0 \text{ SLM} \Rightarrow (E) + (L) + [(L^+) \Rightarrow (J)] + (W)$$

Suff. cond.

$$(E) + (L^+) + (J^+) + (W^+) \Rightarrow u_0 \text{ SLM}$$

Road map for suff. cond.

$$\textcircled{1} \quad (E) + (L^+) + (J^+) \Rightarrow (F) \quad (\text{EMBEDDING THEOREM})$$

$$\textcircled{2} \quad (F) \Rightarrow (WRF) \\ \uparrow \\ \text{Weierstrass representation formula}$$

$$\textcircled{3} \quad (WRF) + (W^+) \Rightarrow u_0 \text{ is SLM}$$

(WRF) If $u(x)$ and $u_0(x)$ have same DBC, then

$$F(u) - F(u_0) = \int_a^b E(\dots) \\ \uparrow \\ \text{suitable points} \\ \text{---} \circ \text{---} \circ \text{---}$$

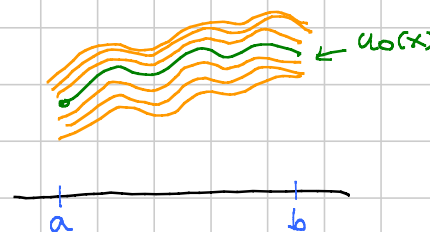
WEIERSTRASS FIELD

Family of solutions to Euler equation which fill an open set around $u_0(x)$.

More precisely, is a function

$$u: (-\varepsilon_0, \varepsilon_0) \times [a, b] \rightarrow \mathbb{R}$$

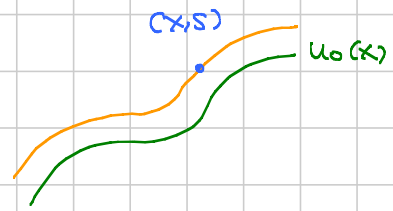
such that $x \mapsto u(\varepsilon, x)$ is a solution to (E) for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and $u_\varepsilon(\varepsilon, x) > 0$ for every (ε, x) admissible



Definition To each WF a SLOPE FUNCTION is associated

Take any (x, s) in the field. There is a unique $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ s.t.

$$u(\varepsilon, x) = s$$



Then the slope function is

$$p(x, s) := \dot{u}(\varepsilon, x) = u_x(\varepsilon, x)$$

The basic relation is

$$u_x(\varepsilon, x) = p(x, u(\varepsilon, x))$$

$$(WRF) \quad F(u) - F(u_0) = \int_a^b E(x, u(x), p(x, u(x)), \dot{u}(x)) dx$$

Calculus of Variations – LECTURE 31

Note Title

23/11/2016

Condition (F) $F(u) = \int_a^b L(x, u, u') dx$

$$L: [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$$

$u_0: [a, b] \rightarrow \mathbb{R}$ solution to Euler eqn.

We say that u_0 satisfies (F) if there exists $\mu: (-\varepsilon_0, \varepsilon_0) \times [a, b] \rightarrow \mathbb{R}$ smooth enough (C^2 is more than enough) such that

(i) $\mu(0, x) = u_0(x) \quad \forall x \in [a, b]$

(ii) $\mu_\varepsilon(\varepsilon, x) > 0 \quad \forall (\varepsilon, x) \in (-\varepsilon_0, \varepsilon_0) \times [a, b]$

(iii) $\forall \varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the function $x \rightarrow \mu(\varepsilon, x)$ is a solution to Euler eqn.

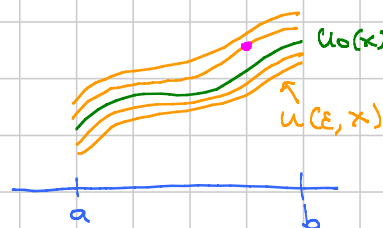
Let Ω be the image of μ ($\Omega \subseteq \mathbb{R}^2$)

The slope function

$$p: \Omega \rightarrow \mathbb{R}$$

defined in such a way that

$$p(x, \mu(\varepsilon, x)) = \mu_x(\varepsilon, x)$$



$$p(x, s)$$

$$(x, s) \in \Omega$$

(WRF) For every $v: [a, b] \rightarrow \mathbb{R}$ with $(x, v(x)) \in \Omega$ for every $x \in [a, b]$ and

$$v(a) = u_0(a)$$

$$v(b) = u_0(b)$$

It turns out that

$$F(v) - F(u_0) = \int_a^b E(x, v(x), p(x, v(x)), v'(x)) dx$$

where $E(x, s, p, q) := L(x, s, q) - L(x, s, p) - (q - p)L_p(x, s, p)$

MAIN RESULT $(F) \Rightarrow (WRF)$

Two strategies : $\rightarrow$ à la Hilbert (via null Lagrangian)
 $\rightarrow$ à la Weierstrass.
 — o — o —

Proof à la Hilbert

First step: rewriting of Euler equ. in terms of slope function

$$p(x, u(\varepsilon, x)) = u_x(\varepsilon, x)$$

let us derive wrt x :

$$p_x(x, u(\varepsilon, x)) + p_s(x, u(\varepsilon, x)) u_x(\varepsilon, x) = u_{xx}(\varepsilon, x)$$

$$p_x(x, u(\varepsilon, x)) + p_s(x, u(\varepsilon, x)) p(x, u(\varepsilon, x)) = u_{xx}(\varepsilon, x)$$

$$p_x + p_s p = u_{xx}$$

Euler equ.: $\frac{d}{dx} L_p(x, u, u) = L_s(x, u, u)$

$$L_{px}(x, u(\varepsilon, x), u_x(\varepsilon, x)) + L_{ps}(x, u(\varepsilon, x), u_x(\varepsilon, x)) u_x(\varepsilon, x) + \\ + L_{pp}(\dots) \underline{u_{xx}(\varepsilon, x)} = L_s(\dots)$$

$$L_{px}(x, u(\varepsilon, x), p(x, u(\varepsilon, x))) + L_{ps}(x, u(\varepsilon, x), p(x, u(\varepsilon, x))) p(x, u(\varepsilon, x)) \\ + L_{pp}(\dots) (p_x + p_s p) = L_s(\dots)$$

$$L_{px}(x, s, p(x, s)) + L_{ps}(x, s, p(x, s)) p(x, s) + \\ L_{pp}(\dots) (p_x + p_s p) = L_s(\dots)$$

$$\boxed{L_{px} + L_{ps} p + L_{pp} p_x + L_{pp} p_s p = L_s} \quad \forall (x, s) \in \Omega.$$

$\begin{matrix} & & (x, s) & & \\ & \downarrow & & \downarrow & \downarrow \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ (x, s, p(x, s)) & & & & \end{matrix}$

Second step: rewriting of (WRF).

$$F(v) - F(u_0) = \int_a^b L(x, v, \dot{v}) - \int_a^b L(x, u_0, \dot{u}_0)$$

$$\int_a^b E(x, v, p(x, v), \dot{v}) = \int_a^b L(x, v, \dot{v}) - \int_a^b L(x, v, p(x, v)) - \int_a^b (\dot{v} - p(x, v)) L_p(x, v, p(x, v))$$

$$(WRF) \Leftrightarrow \underbrace{\int_a^b L(x, u_0, \dot{u}_0) - L(x, v, p(x, v)) - (\dot{v} - p(x, v)) L_p(x, v, p(x, v))}_{\hat{F}(v)} = 0$$

$$\hat{F}(v) = \int_a^b \hat{L}(x, v, \dot{v}) dx \quad \text{where}$$

$$\hat{L}(x, s, p) := \underbrace{L(x, u_0(x), \dot{u}_0(x)) - L(x, s, p(s, x))}_{B(x, s)} + \underbrace{p(x, s) L_p(x, s, p(s, x))}_{A(x, s)}$$

"Clearly" it turns out that $\hat{F}(u_0) = 0$ ($\dot{u}_0(x) = p(x, u_0(x))$)

Therefore (WRF) is proven if we show that $\hat{L}$ is a null Lagrangian, namely if

$$A_x = B_s$$

Third step: $\hat{L}$ is a null Lagrangian

$$A_x = -L_{px} - L_{pp} p_x$$

$$B_s = -L_s - L_{ps} + p L_{ps} + p L_{pp} p_s$$

$$A_x = B_s \Leftrightarrow \text{Euler equ. w.r.t slope function.}$$

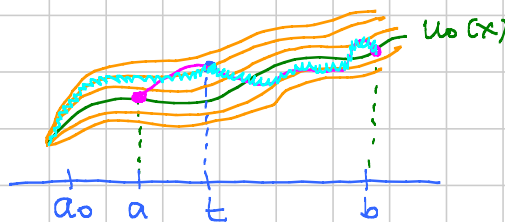
PROOF À LA WEIERSTRASS

In the special case where the field originates from some point $a_0 < a$

Take v as before.

Take any point $t \in [a, b]$.

Consider $\varepsilon(t)$ in such a way that



$$u(\varepsilon(t), t) = v(t)$$

Consider $w(x)$ defined as
$$\begin{cases} u(\varepsilon(t), x) & x \in [a_0, t] \\ v(x) & x \in [t, b] \end{cases}$$

and define

$$f(t) := \int_{a_0}^b L(x, w, \dot{w}) dx$$

$$f(a) = \int_{a_0}^a L(x, u_0, \dot{u}_0) + \int_a^b L(x, v, \dot{v}) dx$$

$$f(b) = \int_{a_0}^a L(x, u_0, \dot{u}_0) + \int_a^b L(x, u_0, \dot{u}_0) dx$$

Therefore

$$F(v) - F(u_0) = f(a) - f(b) = - \int_a^b f'(x) dx$$

Claim: $-f'(x) = E(x, v(x), p(x, v(x)), \dot{v}(x))$

$$f(t) = \int_{a_0}^t L(x, u(\varepsilon(t), x), u_x(\varepsilon(t), x)) dx + \int_t^b L(x, v(x), \dot{v}(x))$$

$f'(t)$ has many terms ...

$$-L(t, v(t), \dot{v}(t)) \quad (\text{second integral})$$

$$L(t, u(\varepsilon(t), t), u_x(\varepsilon(t), t)) = L(t, v(t), p(t, v(t)))$$

$$\underbrace{\int_{a_0}^t L_s(\dots) u_\varepsilon \cdot \varepsilon_t}_{\text{integrate by parts}} + \int_{a_0}^t L_p(\dots) u_{x\varepsilon} \varepsilon_t dx$$

$$= - \underbrace{\int_{a_0}^t \varepsilon_t \frac{d}{dx} L_p(\dots) u_\varepsilon}_{\text{by Euler eqn.}} = 0$$

$$+ \left[L_p(\dots) u_\varepsilon \varepsilon_t \right]_{a_0}^t$$

when $x = a_0 \rightsquigarrow 0$ because $u_\varepsilon(\varepsilon, a_0) = 0$

It remains only $x = t$:

$$L_p(t, u(\varepsilon(t), t), u_x(\varepsilon(t), t)) \underbrace{u_\varepsilon(\varepsilon(t), t) \varepsilon_t(t)}_{\substack{L_p(t, v(t), p(t, v(t))) \\ (\dot{v} - p'(t, v(t)))}}$$

Recall that

$$\begin{aligned} u(\varepsilon(t), t) &= v(t) \\ u_\varepsilon \cdot \varepsilon_t + u_x &= \dot{v} \quad \leadsto \quad u_\varepsilon \cdot \varepsilon_t = \dot{v} - u_x \\ &= \dot{v} - p'(t, v(t)) \end{aligned}$$

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Calculus of Variations

LECTURE 32

Note Title

23/11/2016

WEIERSTRASS NECESSARY CONDITION

$$\boxed{\text{No SLM}} \Rightarrow \boxed{(E) + (L) + [(L^+) \Rightarrow (J)] + (W)}$$

$\Downarrow$ $\nearrow$
 no WLM

Remains to prove $\text{No SLM} \Rightarrow (W)$

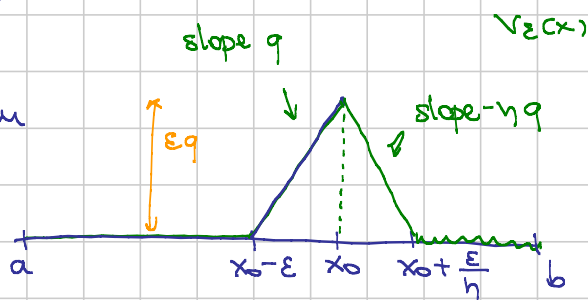
$$(W) \quad E(x, u_0(x), \dot{u}_0(x), q) \geq 0 \quad \forall x \in [a, b], \quad \forall q \in \mathbb{R}$$

Proof. We prove actually that

$$E(x, u_0(x), \dot{u}_0(x), \dot{u}_0(x) + q) \geq 0 \quad \forall x \in (a, b), \quad \forall q \in \mathbb{R}$$

Let $x_0 \in (a, b)$ be any point. Let $q \in \mathbb{R}$. Consider $v_\varepsilon(x)$ as follows:
 Let $\eta > 0$ be any pos. real number

If $\varepsilon > 0$ is small enough, then
 $v_\varepsilon(x)$ is an admissible
 variation (0 at the boundary)
 and therefore



$$F(u_0 + v_\varepsilon) \geq F(u_0) \quad \text{by defn. of SLM}$$

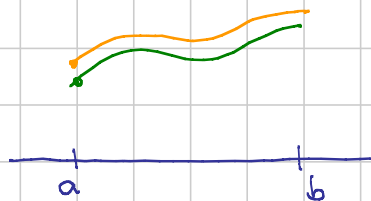
$$F(u_0 + v_\varepsilon) - F(u_0) \geq 0$$

$$\begin{aligned} \text{LHS} &= \int_{x_0 - \varepsilon}^{x_0} L(x, u_0(x) + v_\varepsilon(x), \dot{u}_0(x) + q) - L(x, u_0(x), \dot{u}_0(x)) \, dx \\ &\quad + \int_{x_0}^{x_0 + \frac{\varepsilon}{\eta}} L(x, u_0(x) + v_\varepsilon(x), \dot{u}_0(x) - \eta q) - L(x, u_0(x), \dot{u}_0(x)) \, dx \geq 0 \end{aligned}$$

Let us consider the same equation with initial conditions

$$u(a) = u_0(a) + \varepsilon \alpha$$

$$\dot{u}(a) = \dot{u}_0(a) + \varepsilon \beta$$



Hope: finding α and β in such a way that solutions to the approx. prob. form the required stratification.

Let $u_\varepsilon(x)$ denote the solution. Hope is ok if we show that

$$u_\varepsilon(0, x) \geq m > 0 \quad \text{for every } x \in [a, b]$$

This guarantees that $\exists \varepsilon_0 > 0$ s.t.

$$u_\varepsilon(\varepsilon, x) > 0 \quad \text{for every } (\varepsilon, x) \in (-\infty, \varepsilon_0) \times [a, b]$$

We have reduced the problem to computing the derivative of the solution wrt initial conditions.

Surprise: $u_\varepsilon(0, x) =: w(x)$ is the solution to (JDE) with initial data $w(a) = \alpha$, $\dot{w}(a) = \beta$ and we know that

$(L^+) + (J^+) \Rightarrow (J^{++})$, namely the existence of a solution to (JDE) which is strictly positive in $[a, b]$.

Remains to prove that $w(x)$ solves (JDE)

Sketch of the proof

$$w_\varepsilon(x) := \frac{u(\varepsilon, x) - u(0, x)}{\varepsilon} = \frac{u(\varepsilon, x) - u_0(x)}{\varepsilon}$$

For every $\varepsilon > 0$:

$$[L_p(x, u(\varepsilon, x), \dot{u}(\varepsilon, x))] = L_s(x, u(\varepsilon, x), \dot{u}(\varepsilon, x))$$

$$[L_p(x, u_0(x), \dot{u}_0(x))] = L_s(x, u_0(x), \dot{u}_0(x)) \quad \leftarrow \varepsilon = 0$$

Subtract

$$[L_p(x, u(\varepsilon, x), \dot{u}(\varepsilon, x)) - L_p(x, u_0(x), \dot{u}_0(x))] = L_s(\dots) - L_s(\dots)$$

$$[L_{ps}(\dots)(u(\varepsilon, x) - u_0(x)) + L_{pp}(\dots)(\dot{u}(\varepsilon, x) - \dot{u}_0(x))] =$$

$$L_{ss}(\dots)(u(\varepsilon, x) - u_0(x)) + L_{sp}(\dots)(\dot{u}(\varepsilon, x) - \dot{u}_0(x))$$

Divide by ε :

$$[L_{ps}(\dots) w_\varepsilon + L_{pp}(\dots) \dot{w}_\varepsilon] = L_{ss}(\dots) w_\varepsilon + L_{sp}(\dots) \dot{w}_\varepsilon$$

Take the limit as $\varepsilon \rightarrow 0^+$

$$[L_{ps}(\dots) w + L_{pp}(\dots) \dot{w}] = L_{ss}(\dots) w + L_{sp}(\dots) \dot{w}$$

$\uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow$
 $(x, u_0(x), \dot{u}_0(x))$

$$[B(x) w + C(x) \dot{w}] = A(x) w + B(x) \dot{w} \quad \leftarrow \text{(JDE)}$$

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Simpler example
$$\begin{cases} \dot{u} = f(t, u) \\ u(0) = \varepsilon \end{cases}$$

Let $u(\varepsilon, x)$ denote the solution. Let $v(x) := u_\varepsilon(0, x)$.

What is the equation solved by v ?

$$\dot{u}(\varepsilon, t) = f(t, u(\varepsilon, t))$$

$$\dot{u}(0, t) = f(t, u(0, t))$$

$$\frac{\dot{u}(\varepsilon, t) - \dot{u}(0, t)}{\varepsilon} = \frac{1}{\varepsilon} [f(t, u(\varepsilon, t)) - f(t, u(0, t))]$$

$$\left[\frac{u(\varepsilon, t) - u(0, t)}{\varepsilon} \right]' = \frac{u(\varepsilon, t) - u(0, t)}{\varepsilon} f_u(t, c(\varepsilon, t))$$

$$\boxed{\dot{v} = f_u(t, u_0) v} \quad v(0) = 1$$

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Calculus of Variations

LECTURE 33

Note Title

25/11/2016

Lagrange Multipliers

Back to calculus : $x_0 \in \mathbb{R}^n$, $r > 0$, $F: B_r(x_0) \rightarrow \mathbb{R}$
 $G: B_r(x_0) \rightarrow \mathbb{R}$

$$V := \{x \in B_r(x_0) : G(x) = 0\}$$

$$\min \{F(x) : x \in V\}$$

Lagrange multipliers Assume F and G of class C^1 in $B_r(x_0)$
 Assume that x_0 is a solution to the min. pbm.

Then at least one of the following options is true

(i) $\nabla G(x_0) = 0$ (x_0 is a stationary point for $G(x)$)

(ii) $\exists \lambda \in \mathbb{R}$ s.t. x_0 is a stationary point of $\Phi(x) = F(x) - \lambda G(x)$
 $(\nabla F(x_0) = \lambda \nabla G(x_0))$

Remark If $V = \{x \in B_r(x_0) : G_1(x) = \dots = G_k(x) = 0\}$

then the options are

(i) $\exists (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k \setminus \{0\}$ s.t. x_0 is a stationary point of
 $\lambda_1 G_1(x) + \dots + \lambda_k G_k(x)$

(ii) $\exists (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$ s.t. x_0 is a stationary point of

$$\Phi(x) := F(x) - \lambda_1 G_1(x) - \dots - \lambda_k G_k(x)$$

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Back to calculus of variations

Assume that $F(u)$ and $G(u)$ are two functionals defined in some affine space X with reference vector space V .
 let

$$W = \{u \in X : G(u) = 0\}$$

Theorem Assume

$$u_0 \in \operatorname{argmin} \{ F(u) : u \in W \}$$

Then at least one of the following options is true:

- (i) $\delta G(u_0, v) = 0$ for every $v \in V$ (provided that δG exists)
 (ii) $\exists \lambda \in \mathbb{R}$ s.t. $\delta F(u_0, v) = \lambda \delta G(u_0, v)$ for every $v \in V$,
 namely

$$\delta \Phi(u_0, v) = 0 \quad \forall v \in V$$

where

$$\Phi(u) := F(u) - \lambda G(u).$$

Two strategies : $\rightarrow$ via implicit function thm.
 $\rightarrow$ via inverse function thm

Proof 1 If (i) holds true, there is nothing to prove.

So we can assume that $\delta G(u_0, w) \neq 0$ for some $w \in V$.

Define

$$f(t, s) := F(u_0 + tv + sw) \quad v \in V$$

$$g(t, s) := G(u_0 + tv + sw)$$

We have that

$$g_t(0, 0) = \delta G(u_0, v) \\ g(0, 0) = 0$$

$$g_s(0, 0) = \delta G(u_0, w) \neq 0$$

Implicit function theorem $\Rightarrow \exists \delta > 0$ and $s : (-\delta, \delta) \rightarrow \mathbb{R}$
 such that $s(0) = 0$ and

$$g(t, s(t)) = 0 \quad \forall t \in (-\delta, \delta)$$

and
$$s'(t) = - \frac{g_t(t, s(t))}{g_s(t, s(t))} \rightsquigarrow s'(0) = - \frac{\delta G(u_0, v)}{\delta G(u_0, w)}$$

Now consider $\varphi(t) := f(t, s(t)) = F(\underbrace{u_0 + tv + s(t)w}_{\in W})$
 defined for $t \in (-\delta, \delta)$.

We know that $t=0$ is a min point for φ and therefore $\varphi'(0) = 0$

$$\varphi'(t) = f_t(t, s(t)) + f_s(t, s(t)) s'(t)$$

$$\varphi'(0) = f_t(0, 0) + f_s(0, 0) s'(0)$$

$$= \delta F(u_0, v) + \underbrace{\delta F(u_0, w)}_{\substack{0 \\ \lambda}} \left(- \frac{\delta G(u_0, v)}{\delta G(u_0, w)} \right)$$

$$\Rightarrow \delta F(u_0, v) = \lambda \delta G(u_0, v) \quad \forall v \in V$$

Proof 2 Let $f(t, s)$ and $g(t, s)$ be as before. Let $w \in V$ be as before, namely $g_s(0, 0) = \delta G(u_0, w) \neq 0$. Let us consider

$$\varphi(t, s) := (f(t, s) - \min \{F(u) : u \in W\}, g(t, s))$$

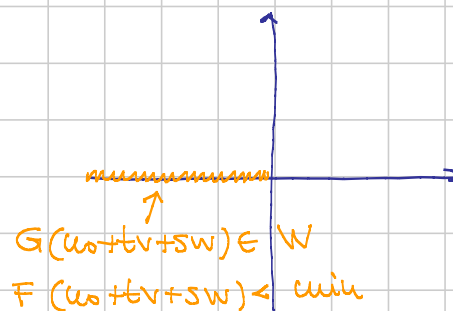
$$\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

Main point: the image of φ does not contain the negative x axis.

Let us consider the matrix

$$J\varphi(0, 0) = \begin{pmatrix} f_t(0, 0) & f_s(0, 0) \\ g_t(0, 0) & g_s(0, 0) \end{pmatrix}$$

$$= \begin{pmatrix} \delta F(u_0, v) & \delta F(u_0, w) \\ \delta G(u_0, v) & \delta G(u_0, w) \end{pmatrix} \quad \varphi(0, 0) = 0$$



Assume that $\text{rank } J\varphi(0, 0) = 2$. Then by the inverse function

then the image of φ contains a neighborhood of the origin, which is absurd.

Therefore $\text{rank} = 1$, and hence $\text{Det} = 0$

$$\delta F(u_0, v) \underbrace{\delta G(u_0, w)}_{\text{--- } 0 \text{ --- } 0 \text{ ---}} = \delta F(u_0, w) \delta G(u_0, v)$$

Rank (exercise) Generalize the 2nd proof to the case of k constraints.

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Example 1

$$F(u) = \int_0^l \dot{u}^2 dx \quad G(u) = \int_0^l u^2 dx - 1$$

$$X = \{u \in C^1([0, l]) : u(0) = u(l) = 0\}$$

Direct method $\Rightarrow$ existence of a min point in $H^1([0, l])$
(only need to check that the constraint is stable with the notion of converge)

By Lagrange method there are 2 possibilities

$$(i) \quad \delta G(u_0, v) = 0 \leadsto \text{Euler eqn of } G \leadsto 0 = 2u \quad (L_p)' = L_s$$

$$\Rightarrow u_0 \equiv 0 \Rightarrow \text{does not satisfy } G(u) = 0$$

(ii) u_0 satisfies the Euler equation of the functional with

$$L(x, s, p) = p^2 - \lambda s^2 \quad (F - \lambda G)$$

$$(2\dot{u})' = -2\lambda u \leadsto \begin{cases} \ddot{u} = -\lambda u \\ u(0) = 0 \\ u(l) = 0 \end{cases} \quad \text{Lagrange multiplier}$$

What about sign of λ ? Multiply by u :

$$\ddot{u}u = -\lambda u^2 \quad \leadsto \text{integrate}$$

$$\int \ddot{u}u = -\lambda \int u^2 = -\lambda$$

$$= -\int \dot{u}^2 + [\dot{u}u]_0^l \quad \Rightarrow \lambda > 0$$

We can solve the equation and obtain

$$u(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

$\overset{0}{0}$ because

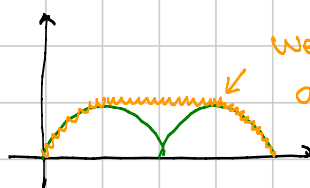
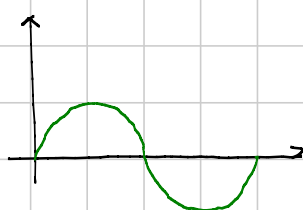
$$u(0) = 0$$

$$\sin(\sqrt{\lambda}l) = 0 \leadsto \sqrt{\lambda} = \frac{k\pi}{l}$$

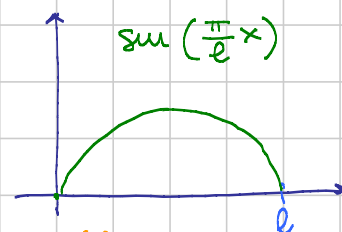
$$u(x) = c \sin\left(\frac{k\pi x}{l}\right) \leadsto \text{use } \int u^2 = 1 \text{ and find } c$$

$\leadsto$ compute $\int \dot{u}^2$ and find that the minimum is for $k=1$

Remark Is it possible to see that $k=1$ is better than $k > 1$?



we save on $\int \dot{u}^2$ and $\int u^2$ is larger, so that we can multiply by a constant less than 1



Calculus of Variations — LECTURE 34

Note Title

25/11/2016

Integral functionals with multiple integralsDomain $\Omega \subseteq \mathbb{R}^m$ (ball or square in $\mathbb{R}^2$). The unknown

$$u: \Omega \rightarrow \mathbb{R} \quad (m \geq 1)$$

$$F(u) := \int_{\Omega} L(x, u(x), \nabla u(x)) dx$$

↑ multiple integral

$$L: \underbrace{\Omega}_{\mathcal{X}} \times \underbrace{\mathbb{R}}_{\mathcal{S}} \times \underbrace{\mathbb{R}^m}_{\mathcal{P}}$$

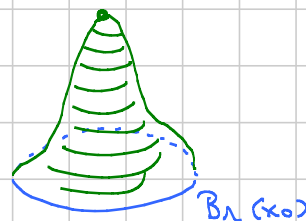
Example (Dirichlet integral) $F(u) = \int_{\Omega} |\nabla u(x)|^2 dx$

Boundary conditions: prescribing the values of $u(x)$ for $x \in \partial\Omega$ Remark Pointwise conditions such as $u(a) = A$ with $a \in \bar{\Omega}$ make no sense for $F(u)$.Roughly speaking, this is because $\forall B_r(x_0)$

$$\inf \{ F(u) : u(x_0) = 1, u(x) \equiv 0 \text{ at } \partial B_r(x_0) \} = 0$$

$$u \in C^1(B_r(x_0))$$

(Exercise: find "optimal profiles")

If $\Omega \subseteq \mathbb{R}^m$, the only conditions which make sense are on sets of dim $m-1$, for example $\partial\Omega$.

— o — o —

$$\min \left\{ \int_{\Omega} |\nabla u|^2 dx : u(x) = \varphi(x) \quad \forall x \in \partial\Omega \right\}$$

given

Indirect method Assume $u_0 \in \text{argmin}$

$$F(u_0 + tv)$$

$\uparrow v \in C^1(\Omega)$ and $v \equiv 0$ at $\partial\Omega$



$$\begin{aligned} &= \int_{\Omega} |\nabla(u_0 + tv)|^2 = \int_{\Omega} |\nabla u_0 + t \nabla v|^2 \\ &= \int_{\Omega} |\nabla u_0|^2 + t^2 \int_{\Omega} |\nabla v|^2 + 2t \int_{\Omega} \langle \nabla u_0, \nabla v \rangle \end{aligned}$$

$$\delta F(u_0, v) = 2 \int_{\Omega} \langle \nabla u_0, \nabla v \rangle dx$$

First form of Euler Equation

In order to integrate by parts, we need Gauss - Green formula

$$\int_{\Omega} f \operatorname{div} \vec{E} = \underbrace{\int_{\partial\Omega} \langle \vec{E}, \vec{n} \rangle f d\sigma}_{(n-1)\text{-dim. integral}} - \int_{\Omega} \langle \nabla f, \vec{E} \rangle dx$$

Let us apply GG with $f := v$ and $\vec{E} = \nabla u_0$. Obtain that

$$\begin{aligned} \int_{\Omega} \underbrace{\langle \nabla v, \nabla u_0 \rangle}_{\substack{\nabla f \quad E \\ \downarrow \quad \downarrow}} dx &= - \int_{\Omega} \underbrace{v \operatorname{div}(\nabla u_0)}_{\substack{f \quad \operatorname{div} E \\ \downarrow \quad \downarrow}} + \underbrace{\int_{\partial\Omega} v \langle \nabla u_0, \vec{n} \rangle d\sigma}_{\substack{f \quad \nabla E \\ \downarrow \quad \downarrow}} \\ &= - \int_{\Omega} \Delta u_0 \cdot v \quad \quad \quad = 0 \text{ because } v \equiv 0 \text{ at } \partial\Omega \end{aligned}$$

where $\Delta u = u_{xx} + u_{yy}$

Therefore, the second form of Euler equ is

$$\boxed{\int_{\Omega} \Delta u_0 \cdot v \, dx = 0} \quad \forall v \text{ s.t. } v \equiv 0 \text{ at } \partial\Omega$$

$$\text{FLCV} \Rightarrow \begin{cases} \Delta u_0 = 0 & (\text{partial differential equ}) \\ u_0 = \varphi & \text{at } \partial\Omega \quad (\text{DBC}) \end{cases} \quad \left. \vphantom{\begin{cases} \Delta u_0 = 0 \\ u_0 = \varphi \end{cases}} \right\} \begin{array}{l} \text{Euler equ.} \\ \text{in diff.} \\ \text{form} \end{array}$$

Assume we are able to solve. Then u_0 is the unique minimum point (by convexity)

Any other competitor w can be written as $u_0 + v$ with $v \in V$ and

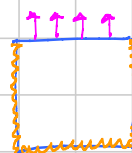
$$F(w) = F(u_0 + v) = \underbrace{\int_{\Omega} |\nabla u_0|^2}_{F(u_0)} + \underbrace{\int_{\Omega} |\nabla v|^2}_{\geq 0} + 2 \underbrace{\int_{\Omega} \langle \nabla u_0, \nabla v \rangle}_{=0 \text{ because of Euler equ.}}$$

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 $\Rightarrow v \equiv 0$

Example 2 $F(u) = \text{Dirichlet integral}$ $\Omega = \text{square}$

$$\min \{ F(u) : u = \varphi \text{ on three sides} \}$$

$\uparrow$
given



As before $\delta F(u_0, v) = \int_{\Omega} \langle \nabla u_0, \nabla v \rangle = 0 \quad \forall v \equiv 0 \text{ on 3 sides}$

Integrate by parts (G.G.):

$$\int_{\Omega} \langle \nabla u_0, \nabla v \rangle \, dx = - \int_{\Omega} \Delta u_0 \cdot v \, dx + \int_{\partial\Omega} v \langle \nabla u_0, \vec{n} \rangle \, d\sigma$$

$\uparrow$
outer normal

1st step We limit to $v \equiv 0$ on the whole $\partial\Omega$
 $\leadsto \Delta u_0 = 0$ in Ω

2nd step Now we know that

$$\int_{\partial\Omega} v \langle \nabla u_0, \vec{n} \rangle d\sigma = 0 \quad \forall v \text{ admissible}$$

$\uparrow$ actually on 3 side only

FLCV on the side and deduce that $\langle \nabla u_0, \vec{n} \rangle = 0$ on the side # 4

$$\langle \nabla u, \vec{n} \rangle = \frac{\partial u}{\partial n} \uparrow \text{directional derivative of } u \text{ in the direction } \vec{n}$$

At the end u_0 has to solve

$$\begin{cases} \Delta u = 0 & \text{PDE} \\ u \equiv \varphi & \text{on 3 sides} & \text{DBC} \\ \frac{\partial u}{\partial n} = 0 & \text{at 4th side} & \text{NBC} \end{cases}$$

What about the general case

$$F(u) = \int_{\Omega} L(x, u, \nabla u) dx$$

$$F(u+tv) =: \varphi(t)$$

$$\varphi'(0) = \frac{d}{dt} \int_{\Omega} L(x, u+tv, \nabla u + t\nabla v) dx \Big|_{t=0}$$

$$= \int_{\Omega} L_s(x, u, \nabla u) v + \langle \nabla_p L(x, u, \nabla u), \nabla v \rangle = \delta F(u, v)$$

$\uparrow$
gradient of L
wrt p variable

The second term can be integrated by parts ...

$$\int_{\Omega} L_s(x, u, \nabla u) v - \operatorname{div} \nabla_p L(x, u, \nabla u) \cdot v + \int_{\partial\Omega} v \langle \nabla_p L(x, u, \nabla u), \vec{n} \rangle d\sigma$$

As before, we restrict to $v \equiv 0$ at $\partial\Omega$

$$\operatorname{div} (\nabla_p L(x, u, \nabla u)) = L_s(x, u, \nabla u)$$

Euler equ. or diff. form

$$\langle \nabla_p L(x, u, \nabla u), \vec{n} \rangle = 0$$

in the points of $\partial\Omega$ where DBC where not prescribed.

— o — o —

What about the direct method?

Sobolev spaces can be defined exactly as in dim 1 (W-H)

In dim. $n \geq 2$ Sobolev functions are not nec. continuous, and in particular H^1 functions are not continuous.

— o — o —

Calculus of Variations — LECTURE 35

Note Title

30/11/2016

Motivational example

$$\min \left\{ \int_0^1 (i^4 + u^8) dx : u(0) = u(1) = 2016 \right\}$$

Main issues : $\rightarrow$ non quadratic growth wrt p
 $\rightarrow L_{pp} \geq 0$ but not strictly positive (affects regularity)

Direct method : let u_n be a min. sequence. We know that $\|i_n\|_{L^4((0,1))}$ is bounded. We would like to deduce that

$$\begin{aligned} u_n &\rightarrow u_\infty \\ i_n &\rightarrow i_\infty \end{aligned}$$

in some sense (up to subsequences)

Remark $\|i_n\|_{L^4} \leq M \Rightarrow \|i_n\|_{L^2} \leq M_1$

Proof. $x^2 \leq x^4 + 1 \quad \forall x \in \mathbb{R}$ and therefore

$$\int_0^1 i_n^2 \leq \int_0^1 i_n^4 + \int_0^1 1$$

Therefore $i_n \rightarrow v_\infty$ in $L^2((0,1))$ (up to subsequences)

Combining with DBC, we obtain that $u_n \rightarrow u_\infty$ unif.
 and $v_\infty = i_\infty$.

Questions : ① $v_\infty \in L^4((0,1))$?

$$\textcircled{2} \liminf_{n \rightarrow \infty} \int_0^1 i_n^4 \geq \int_0^1 v_\infty^4 \quad ?$$

— o — o —

Def. (weak convergence in L^p). Let $\{u_n\} \subseteq L^p(a,b)$ and let $u_\infty \in L^p(a,b)$, with $p \in [1, +\infty)$.

We say that

$$u_n \rightharpoonup u_\infty \quad \text{in } L^p(a,b)$$

if

$$\int_a^b u_n(x) \varphi(x) dx \rightarrow \int_a^b u_\infty(x) \varphi(x) dx \quad \forall \varphi \in L^q(a,b)$$

where $\frac{1}{p} + \frac{1}{q} = 1$ ($q = +\infty$ if $p = 1$).

Remark Since $L^\infty(a,b) \subseteq L^q(a,b)$ for every admissible q , then weak convergence in $L^1(a,b)$ is the weakest. Weak convergence is easier when p is smaller.

Theorem (LSC) Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be a convex function. Let $v_n \rightharpoonup v_\infty$ in $L^1(a,b)$ (weakest possible sense).

Then

$$\liminf_{n \rightarrow +\infty} \int_a^b \psi(v_n) dx \geq \int_a^b \psi(v_\infty) dx$$

Proof in the easy case $\psi \in C^1(\mathbb{R})$ and ψ' bounded. Use the inequality

$$\psi(b) \geq \psi(a) + (b-a) \psi'(a) \quad \forall (a,b) \in \mathbb{R}^2$$

and obtain that

$$\int_a^b \psi(v_n) \geq \int_a^b \psi(v_\infty) + \underbrace{\int_a^b (v_n - v_\infty) \psi'(v_\infty)}_{\substack{\downarrow 0 \text{ as } n \rightarrow +\infty \text{ because } \\ v_n \rightharpoonup v_\infty \text{ in } L^1}} \quad \psi'(v_\infty) \in L^\infty(a,b)$$

Take liminf on both sides.

Lemma (Approx. of convex functions from below)

Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be a convex function.

Then there exists a family of functions $\psi_\varepsilon: \mathbb{R} \rightarrow \mathbb{R}$ s.t.

(i) ψ_ε is convex for every $\varepsilon > 0$

(ii) $\psi_\varepsilon \in C^1(\mathbb{R})$ " " " and ψ'_ε is bounded

(iii) $\psi_\varepsilon(x) \leq \psi(x)$ " " " and every $x \in \mathbb{R}$

(iv)

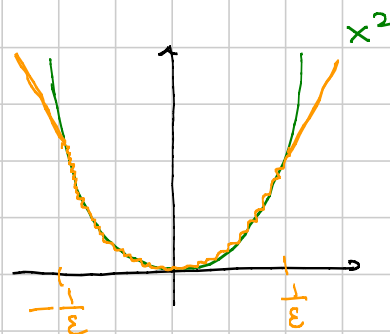
$$\lim_{\varepsilon \rightarrow 0^+} \psi_\varepsilon(x) = \psi(x) \quad \forall x \in \mathbb{R}$$

(v) the family is increasing, namely

$$\psi_{\varepsilon_1}(x) \leq \psi_{\varepsilon_2}(x) \quad \forall x \in \mathbb{R} \quad \forall 0 < \varepsilon_2 \leq \varepsilon_1$$

Proof. Very easy if $\psi \in C^1$, in which case it is enough to truncate ψ' outside $[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]$

If $\psi \notin C^1$, we need to regularize ψ in a first step.



Proof of LSC given the Lemma

Fix $\varepsilon > 0$. For every $n \in \mathbb{N}$:

$$\int_a^b \psi(v_n) dx \geq \int_a^b \psi_\varepsilon(v_n) dx$$

↑
(ccc)

From the easy part we deduce that

$$\liminf_{n \rightarrow \infty} \int_a^b \psi(v_n) dx \geq \int_a^b \psi_\varepsilon(v_\infty) dx$$

Since ψ_ε is an increasing family, from Beppo Levi theorem we deduce that

$$\int_a^b \psi(v_\infty) dx = \sup_{\varepsilon > 0} \int_a^b \psi_\varepsilon(v_\infty) dx.$$

— 0 — 0 —

Back to motivating example $i_n \rightarrow i_\infty$ in L^2 weak
 $u_n \rightarrow u_\infty$ unif.

Therefore $i_n \rightarrow i_\infty$ in L^2 weak

LSC Theorem $\Rightarrow$

$$\liminf_{n \rightarrow \infty} \int_0^1 i_n^4 + u_n^8 \geq \int_0^1 i_\infty^4 + u_\infty^8$$

Direct.

Direct method $\Rightarrow$ existence of minimizers.
 — 0 — 0 —

Growth conditions Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$.

We say that ψ has SUPERLINEAR GROWTH if

$$\lim_{|p| \rightarrow \infty} \frac{\psi(p)}{|p|} = +\infty$$

We say that ψ has (at least) quadratic growth if

$$\liminf_{|p| \rightarrow \infty} \frac{\psi(p)}{p^2} > 0$$

or equivalently if there exist $A > 0$ and $B \in \mathbb{R}$ s.t.

$$\psi(p) \geq A|p|^2 - B \quad \forall p \in \mathbb{R}.$$

Theorem (Compactness) Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be a function with superlinear growth. Let $v_n: [a, b] \rightarrow \mathbb{R}$ be a sequence s.t.

$$\int_a^b \psi(v_n) dx \leq M \leftarrow \text{indep. of } n$$

Then there exist $v_\infty \in L^1((a, b))$ and $n_k \nearrow$ s.t.

$$v_{n_k} \rightharpoonup v_\infty \text{ weakly in } L^1((a, b))$$

Proof in the special case of quadratic growth

We know that

$$\psi(p) \geq Ap^2 - B \quad \forall p \in \mathbb{R}$$

and therefore

$$p^2 \leq \frac{1}{A}(\psi(p) + B)$$

so that

$$\begin{aligned} \int_a^b v_m^2 &\leq \frac{1}{A} \int_a^b \psi(v_m) + \frac{B}{A} (b-a) \\ &\leq M_1 \end{aligned}$$

Up to subsequences $v_m \rightarrow v_\infty$ in L^2 weak, and therefore also in L^1 weak.

Remark $\psi(p) = |p| \log(1+|p|)$ has superlinear growth.

Prop. Assume that $\|v_m\|_{L^p} \leq M$ for some $p > 1$.

Assume that

$$v_m \rightarrow v_\infty \text{ in } L^1((a,b))$$

Then $v_\infty \in L^p$ and

$$v_m \rightarrow v_\infty \text{ in } L^p((a,b))$$

Proof. $v_\infty \in L^p$ follows from the LSC result.

let $\varphi \in L^q((a,b))$, and let $\varepsilon > 0$. Take any function $\varphi_\varepsilon \in L^\infty$ s.t.

$$\|\varphi - \varphi_\varepsilon\|_{L^q} \leq \varepsilon$$

Then

$$\int_a^b (v_m - v_\infty) \varphi = \int_a^b (v_m - v_\infty) \varphi_\varepsilon + \int_a^b (v_m - v_\infty) (\varphi - \varphi_\varepsilon)$$

and therefore

$$\begin{aligned}
 \left| \int_a^b (v_n - v_\infty) \varphi \right| &\leq \underbrace{\left| \int_a^b (v_n - v_\infty) \varphi_\varepsilon \right|}_{\downarrow 0} + \underbrace{\int_a^b |v_n - v_\infty| \cdot |\varphi - \varphi_\varepsilon|}_{\text{Hölder's inequality}} \\
 &\leq \|v_n - v_\infty\|_{L^p} \|\varphi - \varphi_\varepsilon\|_{L^q} \\
 &\leq \underbrace{\|v_\infty\| + \|v_n\|}_{\leq 2M} \underbrace{\|\varphi - \varphi_\varepsilon\|_{L^q}}_{\leq \varepsilon} \\
 &\leq 2M\varepsilon
 \end{aligned}$$

Therefore

$$\limsup_{n \rightarrow \infty} \left| \int_a^b (v_n - v_\infty) \varphi \right| \leq 2M\varepsilon$$

Since ε is arbitrary the limit of the LHS is 0.

— 0 — 0 —

Remark when $\{v_n\}$ is bounded in L^p , it is enough to test weak convergence against a dense subset of test functions of φ .

— 0 — 0 —

Calculus of Variations — LECTURE 36

Note Title

30/11/2016

$$F(u) = \int_a^b L(x, u, \dot{u}) dx$$

Philosophy : $\rightarrow$ growth assumption on L wrt p , for example

$$L(x, s, p) \geq \psi(p) \quad \forall (x, s, p) \in [a, b] \times \mathbb{R}^2$$

with $\psi: \mathbb{R} \rightarrow \mathbb{R}$ with superlinear growth

$\Rightarrow$ weak compactness in $L^1(a, b)$ of u_n , where u_n is any sequence with $F(u_n) \leq M$.

$\rightarrow$ As before + some (integral or pointwise) control on $u_n \rightarrow$ (up to subsequences) $u_n \rightarrow u_\infty$ uniformly.

$\rightarrow$ Some sort of convexity of L wrt p , for example

$$L(x, s, p) := \underbrace{\psi(p)}_{\text{convex}} + g(x, s)$$

$\Rightarrow$ LSC.

— o — o —

Back to motivational example

$$F(u) = \int_0^1 \dot{u}^4 + u^8 \quad \text{with DBC } u(0) = u(1) = 2016$$

Direct method $\Rightarrow$ existence of GM.

Derive Euler equation $\varphi(t) = F(u+tv) = \int_0^1 (\dot{u} + t\dot{v})^4 + (u+tv)^8$

and compute

$$\varphi'(0) = 4 \int_0^1 \dot{u}^3 \dot{v} + 8 \int_0^1 u^7 v = \delta F(u, v)$$

Remark This requires switching integral and derivative, which in turn requires some unif. bounds

$$\varphi'(t) = \int_0^1 \underbrace{4(\ddot{u} + t\ddot{v})^3 \ddot{v} + 8(u + tv)^7 v}_{\text{we need a domination of this indep. of } t. \text{ This is possible because we can control } \ddot{u}^3 \text{ in terms of } \ddot{u}^4.}$$

After integration by parts (or the definition of weak derivative) we end up with

$$(4\ddot{u}^3)' = 8u^7$$

$$\boxed{(\ddot{u}^3)' = 2u^7}$$

Formally this means that

$$\underbrace{3\ddot{u}^2 \ddot{u}}_{\text{Lip}} = 2u^7$$

Question : what about the regularity of u ? Can we say that $u \in C^\infty$? Or even $u \in C^2$?

We know that $\ddot{u} \in L^4 \Rightarrow u \in C^0$ (and actually $\frac{3}{4}$ -Hölder cont.)

$$\Rightarrow (\ddot{u}^3)' \in C^0 \Rightarrow \ddot{u}^3 \in C^1 \stackrel{?}{\Rightarrow} \ddot{u} \in C^1$$

$$\ddot{u}^3 \in C^1 \Rightarrow \ddot{u} \in C^0 \Rightarrow u \in C^1$$

$\Downarrow$

$$\ddot{u}^3 \in \text{Lip} \Rightarrow \ddot{u} \in \frac{1}{3}\text{-Hölder continuous}$$

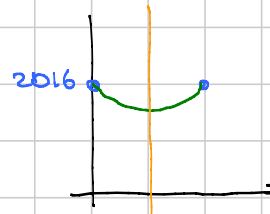
General fact Assume that $f: [a, b] \rightarrow \mathbb{R}$ is continuous and f' is Lipschitz for some exponent $\alpha > 1$. Then f is $\frac{1}{\alpha}$ -Hölder continuous

In conclusion, in the example $u \in C^{\frac{1}{3}, \frac{1}{3}}([0, 1])$
 $\uparrow \quad \uparrow$
 $u \in C^1 \quad \ddot{u} \in \frac{1}{3}\text{-Hölder cont.}$

Remark $u \in C^\infty(A)$ where $A = \{x \in (0,1) : u(x) \neq 0\}$

$$u'' = \frac{2u^*}{3u^3} \text{ in } A \text{ no bootstrap argument.}$$

In the example we are sure that $u'(x) = 0$ somewhere in $(0,1)$ because of DBC.



Exercises (?)

- Prove that $u(x) = u(1-x)$
- Prove that $u \notin C^{1,\alpha}([0,1])$ if $\alpha > \frac{1}{3}$.

Uniqueness is trivial by convexity
 $\quad \quad \quad \underline{\quad} \quad \underline{\quad} \quad \underline{\quad}$

Example

$$\min \left\{ \int_0^2 (1+u^2) u^6 : u(0)=0, u(2)=2016 \right\}$$

$$L(x,s,p) = (1+s^2)p^6$$

$L(x,s,p) \geq p^6$ for every admissible $(x,s,p) \Rightarrow$ compactness

$$F(u_n) \leq M_1 \Rightarrow \|u_n\|_{L^6} \leq M_1 \Rightarrow \begin{aligned} & \text{(up to subsequences)} \\ & u_n \rightarrow u_\infty \text{ in } L^1 \text{ or } L^6 \\ & u_n \rightarrow u_\infty \text{ unif.} \end{aligned}$$

We need LSC

$$F(u_n) = \int_0^1 \underbrace{g(u_n)}_{(1+u_n^2)} \underbrace{\psi(u_n)}_{u_n^6} = \int_0^1 g(u_\infty) \psi(u_\infty) + \int_0^1 (g(u_n) - g(u_\infty)) \psi(u_n)$$

$\uparrow$
 small in L^∞
 norm when u
 is large

As for the first term:

$$\int_0^1 g(u_\infty) \psi(u_n) \geq \int_0^1 g(u_\infty) \psi(u_\infty) + \int_0^1 g(u_\infty) \psi'(u_\infty) (u_n - u_\infty)$$

we used that $\psi(b) \geq \psi(a) + (b-a)\psi'(a)$
and $g(s) \geq 0$

Now we can take the limit on the left and on the RHS.

If we want to be precise, there are two possibilities

→ again approximate ψ with the lemma so that $\psi'(u_n) \in L^\infty$

→ $\psi(u_n) = 6u_n^5 \in L^{\frac{6}{5}}((0,2))$ and $p=6$ so $q=\frac{6}{5}$

$$\text{So } \int_a^b \underbrace{g(u_n) \psi'(u_n)}_{\substack{\text{test function} \\ \text{in } L^{6/5}}} \underbrace{(u_n - u_n)}_{\substack{\rightarrow 0 \\ \text{weakly in } L^6}} \rightarrow 0$$

⇒ once again conclusion by direct method.

Example $\int_0^l u^4 - |u|^\alpha$ with $u(0) = u(1) = 0$

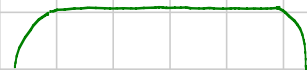
In this case

→ existence by direct method if $\alpha < 4$

→ non-existence (and actually $\inf = -\infty$) if $\alpha > 4$

→ If $\alpha = 4$ we have existence or non-existence depending on l (existence for small l).

— o — o —



Calculus of Variations

LECTURE 37

Note Title

02/12/2016

RELAXATION OF INTEGRAL FUNCTIONALS

Let us consider $F(u) := \int_a^b \psi(u') dx$ when $u \in C^1$ or $u \in C^\infty$.

Goal: extending $F(u)$ to L^2 , L^1 , C^0 through relaxation.

Key words : $\rightarrow$ convexity
 $\rightarrow$ growth assumptions

Theorem (convex case) Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ with

$\rightarrow \psi$ is convex

$\rightarrow \psi$ has quadratic growth at $\pm\infty$ (superlinear growth is enough)

Then the relaxation of $F(u)$ to any L^p or C^0 is

$$F(u) := \int_a^b \psi(u') dx \quad \text{if } u \text{ has a weak derivative } u'$$

$\uparrow$ even this case could give $+\infty$

$$F(u) := +\infty \text{ otherwise}$$

Recall a strategy for proving that some $G(u)$ is the relaxation of $F(u)$

① $F(u) \geq G(u)$ for every u admissible

② $G(u)$ is LSC

③ There exists D dense in energy with respect to G such that for every $u \in D$ there exists $u_n \rightarrow u$ (in the ambient space) and $F(u_n) \rightarrow G(u)$
 (but $\limsup_{n \rightarrow \infty} F(u_n) \leq G(u)$ is enough)

— o — o —

Proof Let $G(u)$ denote what we stated to be the relaxation.

Step ① is trivial because $F(u) = +\infty$ outside C^* or C^∞ .

Step ② is more or less the LSC result of previous lectures.

To be more precise, let $u_n \rightarrow u_\infty$ be any sequence.

We need to prove that

$$\liminf_{n \rightarrow \infty} G(u_n) \geq G(u_\infty)$$

We can assume wlog that $\liminf = \lim < +\infty$. By the growth assumptions this implies that

$$\|u_n\|_{L^2} \leq M \quad \left(\psi(p) \geq \underset{\substack{\uparrow \\ >0}}{Ap^2 - B} \right)$$

So up to subsequences $u_n \rightarrow u_\infty$ in $L^2(a,b)$.

Now we can apply the LSC result (thanks to convexity) and deduce that

$$\liminf_{n \rightarrow \infty} \int_a^b \psi(u_n) dx \geq \int_a^b \psi(u_\infty) dx$$

(just requires $u_n \rightarrow u_\infty$ in L^1)

Step ③.1 A good subset dense in energy wrt G is the set of piecewise affine functions.

For every $u \in L^p, L^1, C^0$ we need $u_n \rightarrow u$

such that $G(u_n) \rightarrow G(u)$ $\uparrow$
piecewise affine

Again $\limsup G(u_n) \leq G(u)$ is enough

Strategy: CONNECT THE DOTS !!

wlog $u \in C^0$ (if not, $G(u) = +\infty$
because $G(u) < +\infty \Rightarrow u \in H^1$
because of quadratic growth)

Simple exercise: $u_n \rightarrow u$ unif.

Moreover: $G(u_n) \leq G(u) \Rightarrow \limsup$ i'eq.

Why $G(u_n) \leq G(u)$? Two possible arguments

→ this is true in each subinterval

$$\int_{x_i}^{x_{i+1}} \psi(\text{line}) dx \leq \int_{x_i}^{x_{i+1}} \psi(u_i) dx$$

$\uparrow$
 $\frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i}$

(The line is the best way to fit BCs in the subinterval)

→ Jensen inequality in each interval (Jensen inequality for integrals)

Step (3.2) Every u which is piecewise affine can be approximated in L^1, L^p, C^0 by a sequence $u_n \rightarrow u$ with $u_n \in C^1$ or C^∞ in such a way that

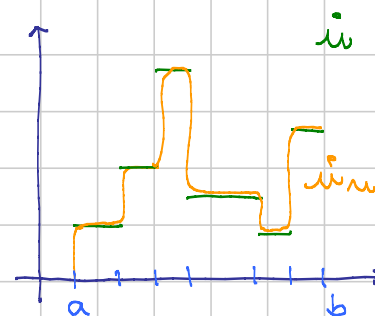
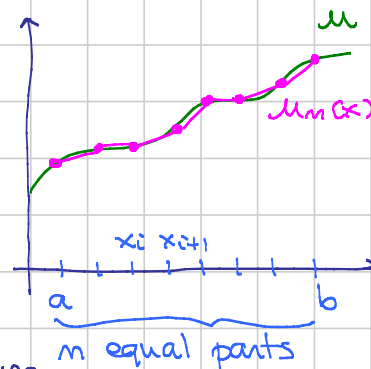
$$\mathbb{F}(u_n) \rightarrow G(u)$$

We can approximate $\dot{u}_n \rightarrow \dot{u}$
 $\in C^0$ or C^∞ in any L^p space

Define

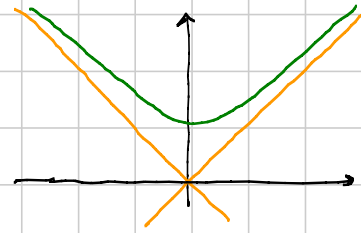
$$u_n(x) := u(a) + \int_a^x \dot{u}_n(t) dt$$

and the proof is complete. $\square$



Remark 1 Things are very different if ψ has just linear growth.

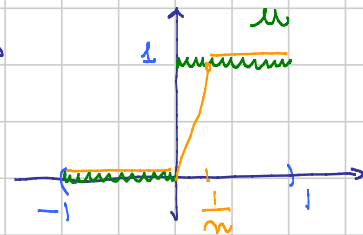
Classical example $\psi(p) := \sqrt{1+p^2}$



In this case there are discontinuous functions u for which $\bar{F}(u) < +\infty$

Can approximate u in any L^p with $p < +\infty$ by a sequence $u_n \in C^1$ with

$$\int_{-1}^1 \psi(u_n) dx \leq M$$



Idea: take u_n as above

$$\int_{-1}^1 \psi(u_n) dx = \int_{-1}^0 1 dx + \int_0^{1/n} 1 dx + \int_{1/n}^{1/n} \sqrt{1+n^2} dx$$

$\underbrace{\qquad}_{=1} \qquad \underbrace{\qquad}_{\leq 1} \qquad \underbrace{\frac{1}{n} \sqrt{1+n^2}}_{\text{bounded}}$

Future extension In this case the relaxation is finite in the BV space
 $\uparrow$
 bounded variation



Remark 2 Let us consider $\psi(p) = e^{-p}$

Problem: Lack of growth as $p \rightarrow +\infty$.

In this case $\bar{F}(u) = \uparrow$ for every nondecreasing function
 the same

Let us consider u as before and $u_n \rightarrow u$ as before

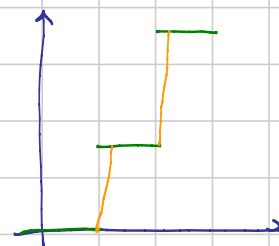
$$\int_{-1}^1 \psi(u_n) dx = \int_{-1}^0 + \int_{1/n}^1 + \underbrace{\frac{1}{n} e^{-n}}_{\downarrow 0} \leq 2$$

The same is true for any function with two jumps in ascending order.

Therefore any increasing piecewise constant function pays less than z .

Conclusion: increasing pwc functions are dense in energy in the set of nondecreasing functions.

— o — o —



Calculus of Variations

LECTURE 38

Note Title

02/12/2016

RELAXATION OF NONCONVEX FUNCTIONALSDef. (Convex envelop of a function)

Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be any function. We call convexification (or convex envelope) of ψ the function

$$\psi^{**}(x) = \sup \{ \varphi(x) : \varphi \text{ is a convex function } : \mathbb{R} \rightarrow \mathbb{R} \text{ with } \varphi(x) \leq \psi(x) \ \forall x \in \mathbb{R} \}$$

Remark Either we allow function φ with values in $\mathbb{R} \cup \{-\infty\}$ or we need to assume the existence of at least one convex $\varphi \leq \psi$.

Prop. (Properties of ψ^{**})

Assume $\exists \varphi \leq \psi$ with φ convex.

Then

(i) ψ^{**} is a convex function

(general fact: the supremum of any family of convex functions is convex because $\text{epigraph} = \bigcap \text{epigraphs}$)

(ii) $\psi^{**}(x) \leq \psi(x)$ for every $x \in \mathbb{R}$,

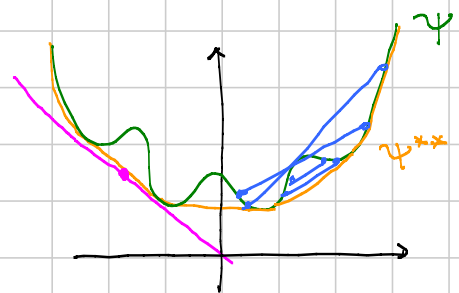
(iii) the sup in the definition is a maximum

(iv) equivalent definition

$$\psi^{**}(x) := \sup \{ ux + n : my + n \leq \psi(y) \text{ for every } y \in \mathbb{R} \}$$

(v) equivalent definition:

$$\psi^{**}(x) := \inf \{ ux + n : \text{line } my + n \text{ intersect graph of } \psi \text{ both before and after } x \}$$



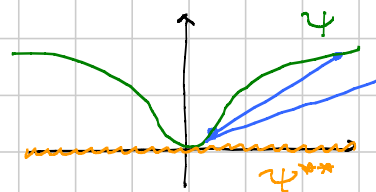
Remark ① Definition (v) is equivalent just in dim 1

② Let us consider

$$\psi(p) := \arctan(p^2)$$

$$\psi(p) := \log(1+p^2)$$

— 0 — 0 —



Theorem (Relaxation in the general case)

Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be any function s.t.

(i) ψ is continuous (maybe not needed)

(ii) ψ has quadratic growth (again superlinear is enough)

Then the relaxation of

$$F(u) := \int_a^b \psi(u) dx$$

$u \in C^1$ or C^∞
(and $+\infty$ otherwise)

is

$$G(u) := \begin{cases} \int_a^b \psi^{**}(u) dx & \text{if } u \in H^1 \\ +\infty & \text{otherwise} \end{cases}$$

(even in this case it might be $+\infty$)

Proof Again step 1, 2 and 3 as before

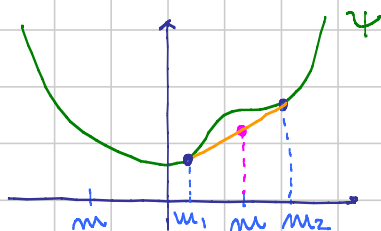
Step ① Trivial because $\psi \geq \psi^{**}$

Step ② Same as before. just used to remark that ψ^{**} has quadratic growth ($\psi(p) \geq Ap^2 - B$
 $\uparrow$ convex and therefore competes in defining ψ^{**})

Step ③.1 As before, the set of p.w. affine functions is dense in energy wrt G .
 (we proved it exploiting the convexity of ψ^{**})

Step (3.2) We need a recovery sequence for every u p.w. affine.

Let us assume first that u is affine: $u(x) = mx + n$



If $\psi(u) = \psi^{**}(u)$, then there is nothing to do: $u_n \equiv u$ does the job.

So wlog $\psi^{**}(u) < \psi(u)$. Assume for a while that there exist u_1 and u_2 such that

$(u, \psi^{**}(u))$ lies on the segment $(u_1, \psi(u_1)), (u_2, \psi(u_2))$

This means that

$$u = \lambda u_1 + (1-\lambda) u_2$$

$$\psi^{**}(u) = \lambda \psi(u_1) + (1-\lambda) \psi(u_2)$$

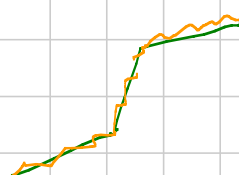
Then we can approximate u by means of a sequence of pw affine functions with slopes $\in \{m_1, m_2\}$

$$F(u) = \lambda(b-a) \psi(u_1) + (1-\lambda)(b-a) \psi(u_2)$$

$$= (b-a) \psi^{**}(u) = \int_a^b \psi^{**}(u) = G(u)$$

If $\psi^{**}(u)$ is not on the segment, then it is in some segment up to a small ε (because of property (v) of ψ^{**}).

If u is just p.w. affine, then we do the same procedure in each subinterval.



Example

$$\inf \left\{ \underbrace{\int_0^{10} (1-u^2)^2 + (u - \cos x)^4 dx}_{F(u)} : u \in C^1([0,1]) \right\}$$

$$\psi(p) := (1-p^2)^2$$

ψ has quadratic growth



$\Rightarrow F(u)$ is coercive ($F(u_n) \leq M \Rightarrow u_p$ to subseq.

$$u_n \rightarrow u_\infty \text{ unif.}$$

$$\dot{u}_n \rightarrow \dot{u}_\infty \text{ weakly in } L^2)$$

The relaxation of $F(u)$ in C^0 or L^{24} is

$$G(u) := \int_0^{10} \psi^{**}(u) + (u - \cos x)^4 dx \quad +\infty \text{ when there are jumps.}$$

(because the second integral is continuous)

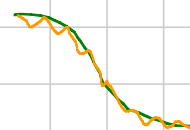
Since F is coercive, then $\inf F = \min G$ because

$u(x) \equiv \cos x$ is a min. point (actually the unique min point)

The point is that $|u'(x)| \leq 1$ and hence $\psi^{**}(u'(x)) \equiv 0$.

Of course it is not a min. for $F(u)$. On the other hand, if u_n is any min. seq. for $F(u)$ (meaning that $F(u_n) \rightarrow \inf$), then

$$u_n(x) \rightarrow \cos x \text{ uniformly}$$



It could be interesting to understand the limit of $\dot{u}_n(x)$

— o — o —

Calculus of Variations

LECTURE 39

Note Title

07/12/2016

GEODESICS Finding the shortest curve between 2 given points.

PLANE (or $\mathbb{R}^n$) $\gamma(t) = (u(t), v(t))$ $\gamma: [a, b] \rightarrow \mathbb{R}^2$

$$\min \left\{ \int_a^b \sqrt{\dot{u}^2 + \dot{v}^2} dt : \gamma(a) = A, \gamma(b) = B \right\}$$

$\mathbb{R}^2$ $\mathbb{R}^2$

$$L(p_1, p_2) := \sqrt{p_1^2 + p_2^2}$$

Bad news: linear growth

Euler equ: $\left(\frac{\dot{u}}{\sqrt{\dot{u}^2 + \dot{v}^2}} \right)' = 0$ $\left(\frac{\dot{v}}{\sqrt{\dot{u}^2 + \dot{v}^2}} \right)' = 0$

$$\frac{\dot{u}}{\sqrt{\dot{u}^2 + \dot{v}^2}} = c_1, \quad \frac{\dot{v}}{\sqrt{\dot{u}^2 + \dot{v}^2}} = c_2$$

After some discussion (the denom. could be $=0$) it is reasonable to expect that

$$\ddot{u}(t) = c \dot{v}(t) \quad \forall t \in [a, b]$$

from which we deduce that $\gamma(t)$ is a parametrization of the segment AB.

Remark Any injective param. of AB is good.
This gives NON UNIQUENESS



Two calibrations by null lagrangians

Calibration 1 Let $B-A = (\alpha, \beta)$. Then

$$\alpha p_1 + \beta p_2 \stackrel{\substack{\uparrow \\ \text{C-S.}}}{\leq} \sqrt{\alpha^2 + \beta^2} \sqrt{p_1^2 + p_2^2}$$

Therefore

$$\begin{aligned} \int_a^b \sqrt{\dot{u}^2 + \dot{v}^2} dt &\geq \frac{1}{\sqrt{\alpha^2 + \beta^2}} \int_a^b \alpha \dot{u} + \beta \dot{v} \\ &= \frac{1}{AB} \left(\alpha \underbrace{[u(t)]_a^b}_{\alpha} + \beta \underbrace{[v(t)]_a^b}_{\beta} \right) = AB \end{aligned}$$

This proves that the segment minimizes the length.

We have equality iff $(\alpha, \beta) = \lambda(\dot{u}, \dot{v}) \leadsto$ Euler equation

Calibration 2 Use that

$$u\ddot{u} + v\ddot{v} \stackrel{\substack{\uparrow \\ \text{C-S.}}}{\leq} \sqrt{u^2 + v^2} \cdot \sqrt{\dot{u}^2 + \dot{v}^2}$$

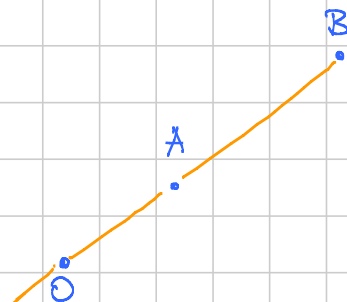
Therefore

$$\begin{aligned} \int_a^b \sqrt{\dot{u}^2 + \dot{v}^2} dt &\geq \int_a^b \left(\frac{u\ddot{u}}{\sqrt{u^2 + v^2}} + \frac{v\ddot{v}}{\sqrt{u^2 + v^2}} \right) dt \\ &= \int_a^b \frac{d}{dt} \sqrt{u^2 + v^2} dt = \left[\sqrt{u^2 + v^2} \right]_a^b \end{aligned}$$

Assume that A, B and the origin lie on the same line

In this case

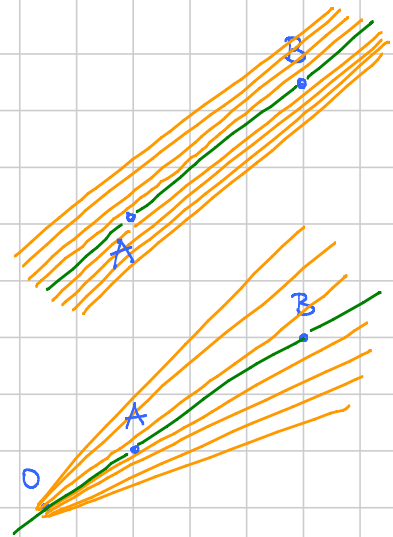
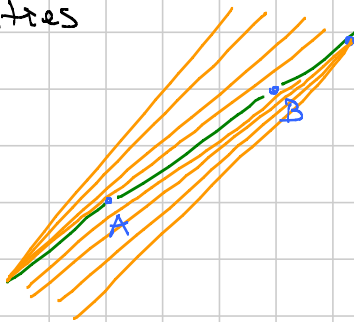
$$\left[\sqrt{u^2 + v^2} \right]_a^b = OB - OA = AB$$



From the geometrical point of view

- Calibration 1 comes from a family of parallel lines
- Calibration 2 comes from the lines originating from 0

Of course there are many other possibilities



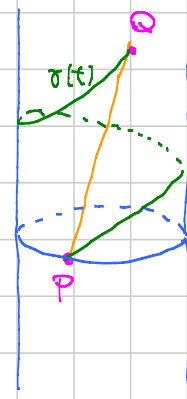
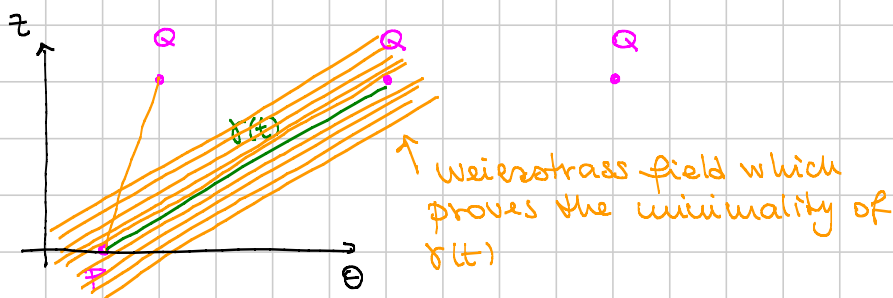
CYLINDER In $\mathbb{R}^3$ $x^2 + y^2 = 1$.

A curve with values in the cylinder is

$$\gamma(t) := (\cos \theta(t), \sin \theta(t), z(t)) \quad t \in [0, 1]$$

and
$$F(\gamma) = \int_0^1 \sqrt{\dot{\theta}^2 + \dot{z}^2} dt$$

Situation is similar to the plane, but



$\gamma(t)$ is a SLM but not a GM. It is the GM among all curves turning once around the cylinder

SPHERE Let us use spherical coordinates

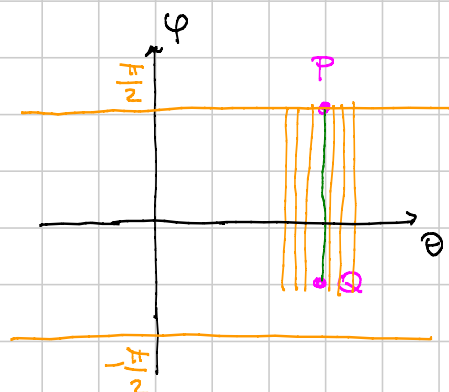
$$\begin{aligned} x &= \rho \cos \varphi \cos \theta & \theta &\in [0, 2\pi] \\ y &= \rho \cos \varphi \sin \theta & \varphi &\in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ z &= \rho \sin \varphi \end{aligned}$$

sphere $\Rightarrow \rho = \text{constant} = 1$ Curve $\leadsto \theta(t), \varphi(t)$

$$\text{length} := \int_a^b \sqrt{\dot{\varphi}^2 + \cos^2 \varphi \dot{\theta}^2} dt$$

We can assume wlog, that $P = \text{North Pole}$

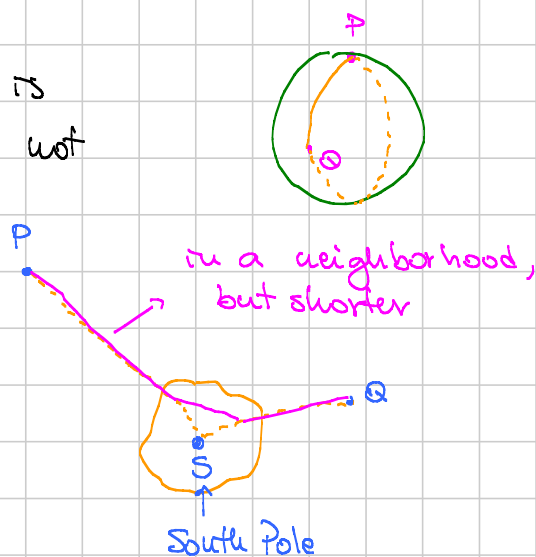
$$\begin{aligned} \int_a^b \sqrt{\dot{\varphi}^2 + \cos^2 \varphi \dot{\theta}^2} dt &\geq \int_a^b \sqrt{\dot{\varphi}^2} \\ &= \int_a^b |\dot{\varphi}| \\ &\geq \left| \int_a^b \dot{\varphi}(t) dt \right| \\ &= |\varphi(P) - \varphi(Q)| \end{aligned}$$



The geodesic is the meridian (the shortest one).

What about the other meridian? It is a solution to Euler equ, but it is not a WLM.

In some sense, the South Pole is the conjugate point of the North Pole.



MIRACLE OF GEODESICS

Let us consider geodesics on any surface

$$\gamma: [a, b] \rightarrow \mathbb{R}^n \quad \text{with} \quad \gamma(t) \in S \quad \forall t \in [a, b]$$

$\uparrow$
given

$$F_1(\gamma) := \int_a^b \|\dot{\gamma}(t)\| dt$$

$$F_2(\gamma) := \int_a^b \|\dot{\gamma}(t)\|^2 dt$$

$\uparrow$
quadratic growth
 $\leadsto$ very good

Miracle The (unique) minimizer of F_2 is also one of the minimizers of $F_1(\gamma)$.

$$F_1(\gamma) = \int_a^b \|\dot{\gamma}(t)\| \cdot 1 dt \leq \underbrace{\left(\int_a^b \|\dot{\gamma}(t)\|^2 dt \right)^{1/2}}_{\text{C.S.}} \cdot \sqrt{b-a}$$

This proves that

$$\min F_1 \leq \sqrt{b-a} (\min F_2)^{1/2}$$

But we have equality if we assume the following.

Lemma (reparametrization Lemma) Given any curve $\gamma: [a, b] \rightarrow \mathbb{R}^n$ there exists a nondecreasing function $\varphi: [a, b] \rightarrow [a, b]$ with $\varphi(a) = a$ and $\varphi(b) = b$ such that $\gamma_1(t) := \gamma(\varphi(t))$ has constant speed, namely $\|\dot{\gamma}_1(t)\| = \text{constant}$ for almost every $t \in [a, b]$.

Remark If $\gamma \in C^1$, then we cannot conclude that $\gamma_1 \in C^1$!!

$\nearrow$ is the image of a C^1 curve, but not the image of C^1 curve with constant speed.

Calculus of Variations

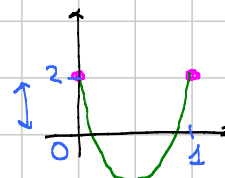
LECTURE 40

Note Title

07/12/2016

OBSTACLE PROBLEMSMotivating example

$$F(u) = \int_0^1 (u^2 + 100u) dx$$

with DBC $u(0) = 2, u(1) = 2.$ Obstacle: $u(x) \geq 0 \quad \forall x \in [0, 1]$ or $u(x) \geq \sin(12x)$ Obstacles on the function: conditions such as

$$u(x) \geq \varphi(x) \quad \forall x \in [a, b]$$

$$\varphi_1(x) \leq u(x) \leq \varphi_2(x) \quad \forall x \in [a, b]$$

$$u(x) \geq \varphi(x) \quad \forall x \in [c, d] \subseteq [a, b]$$

↑
givenObstacles on the derivative: $u'(x) \geq -1 \quad \forall x \in [a, b]$

$$u'(x) \geq \varphi(x) \quad \forall x \in [c, d] \subseteq [a, b].$$

— 0 — 0 —

DIRECT METHODS

- Obstacles on the function pass to the limit w/ unif. convergence $\leadsto$ NO PBM.

$$\min \{ F(u) : \text{DBC} + u(x) \geq \sin(10x) \text{ in } [0, 1] \}$$

Take a min. sequence $u_n \leadsto$ obtain a bound on $\|u_n\|_{L^2}$ $\leadsto$ deduce a bound on $\|u_n\|_\infty$ because of DBC $\leadsto$ apply AA. and up to subsequences $u_n \rightarrow u_\infty$ unif
 $u_n \rightarrow u_\infty$ $\leadsto$ deduce that $u_\infty(x) \geq \sin(10x)$ in $[0, 1]$ $\Rightarrow$ EXISTENCE 😊

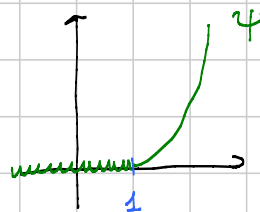
- Obstacles on the derivative are more difficult to deal with
BUT
we can use an integral trick.

Assume $v_n \rightarrow v_\infty$ and $v_n(x) \leq 1$ in $[a, b]$

Can we deduce that $v_\infty(x) \leq 1$ in $[a, b]$? YES

Proof. Take $\psi: \mathbb{R} \rightarrow \mathbb{R}$ as below

- convex
- $= 0$ before 1
- > 0 after 1



For every $n \in \mathbb{N}$ we have that $\int_a^b \psi(v_n) dx = 0$.

Since ψ is convex

$$0 \leq \int_a^b \psi(v_\infty) dx \leq \liminf_{n \rightarrow \infty} \int_a^b \psi(v_n) dx = 0$$

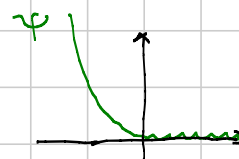
Therefore $\psi(v_\infty) = 0$ almost everywhere, and this means that $v_\infty \leq 1$ almost everywhere.

— o — o —

If the obstacle is $u(x) \geq \sin x$ a similar procedure works starting from

$$\int_a^b \psi(u_n(x) - \sin x) dx$$

— o — o —



Next steps:

- Euler equ.
- regularity

Assume obstacle on the function $u(x) \geq \varphi(x)$

Assume that u is a minimizer for $F(u) = \int_a^b L(x, u, u') dx$

Consider $\varphi(t) := F(u + tv)$ with $v(x) \geq 0$ in $[a, b]$ and $t \geq 0$

By minimality we deduce that $\varphi'(0) \geq 0$



$$\leadsto \int_a^b (L_s(x, u, u')v + L_p(x, u, u')v') \geq 0$$

$$\leadsto \int_a^b (-[L_p(x, u, u')] + L_s(x, u, u'))v \geq 0 \quad \forall v \geq 0$$

Variant of FLCV $\Rightarrow [L_p(x, u, u')] \leq L_s(x, u, u')$

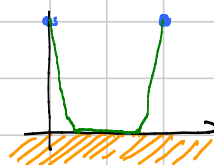
with equality in all points where $u(x) > \varphi(x)$ (in that case we can use any v in a neigh of x).

In conclusion

→ inequality everywhere

→ equality outside the contact region

Back to the example



$$L(x, s, p) = p^2 + 100s$$

$$(2u')' \leq 100$$

↑ at least in weak sense
that $u \in H^1$

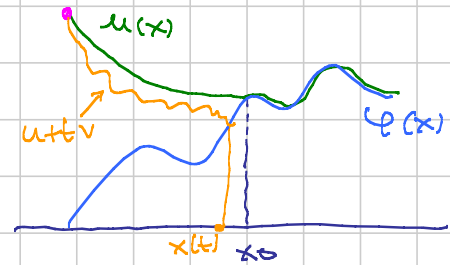
Contact condition

If $L(x, s, p)$ is strictly convex w.r.t p and the obstacle $u(x) \geq \varphi(x)$ is of class C^1 , then in any contact point x_0 it turns out that

$$\dot{u}(x_0) = \dot{\varphi}(x_0)$$

"Proof" Assume a simple situation where

$$\begin{aligned} u(x) &> \varphi(x) & \text{for } x < x_0 \\ u(x) &= \varphi(x) & \text{for } x > x_0 \end{aligned}$$



We claim that

$$L_p(x_0, u(x_0), \dot{u}(x_0)) (\dot{u}(x_0) - \dot{\varphi}(x_0)) = L(x_0, u(x_0), \dot{u}(x_0)) - L(x_0, u(x_0), \dot{\varphi}(x_0))$$

If L is strictly convex, this can be true $\Leftrightarrow \dot{u}(x_0) = \dot{\varphi}(x_0)$

As in the theory of point-to-curve plan we consider a variation $v(x)$, with any sign, we consider $x(t)$ in such a way that

$$u(x(t)) + t v(x(t)) = \varphi(x(t))$$

We compare $F(u)$ with $F(u+tv)$

$$\varphi(t) := \int_a^{x(t)} L(x, u+tv, \dot{u}+t\dot{v}) + \int_{x(t)}^b L(x, \varphi, \dot{\varphi})$$

Standard calculation

$$\begin{aligned} \varphi'(0) &= L(x_0, u(x_0), \dot{u}(x_0)) \dot{x}(0) + \int_a^{x_0} L_s(\dots) v + L_p(\dots) \dot{v} - \\ &\quad - L(x_0, \varphi(x_0), \dot{\varphi}(x_0)) \dot{x}(0) = 0 \end{aligned}$$

$$\dot{x}(0) [L(x_0, u(x_0), \dot{u}(x_0)) - L(x_0, u(x_0), \dot{\varphi}(x_0))] = - \int_a^{x_0} (L_s v + L_p \dot{v})$$

$$\begin{aligned}
&= - \int_a^{x_0} \underbrace{(L_s - [L_p]')}_0 v - \underbrace{[L_p v]_a}_{\substack{\uparrow \\ \text{when } x=a \text{ something vanishes}}} \\
&= L_p(x_0, u(x_0), \dot{u}(x_0)) v(x_0)
\end{aligned}$$

It remains to compute $\dot{x}(0)$:

$$[u(x(t)) + t v(x(t))] = [\varphi(x(t))]'$$

$$\dot{u}(x(t)) \dot{x}(t) + v(x(t)) + t \dot{v}(x(t)) \dot{x}(t) = \dot{\varphi}(x(t)) \dot{x}(t)$$

$$\text{Set } t=0 : \quad \dot{u}(x_0) \dot{x}(0) + v(x_0) = \dot{\varphi}(x_0) \dot{x}(0)$$

$$\leadsto \dot{x}(0) = - \frac{v(x_0)}{\dot{u}(x_0) - \dot{\varphi}(x_0)} \quad \uparrow \text{ if this is } \neq 0.$$

In conclusion

$$- \frac{\cancel{v(x_0)}}{\dot{u}(x_0) - \dot{\varphi}(x_0)} [L(x_0, u, \dot{u}) - L(x_0, \varphi, \dot{\varphi})] = L_p(x_0, u, \dot{u}) \cancel{v(x_0)}$$

Rearranging the terms we obtain the equality.

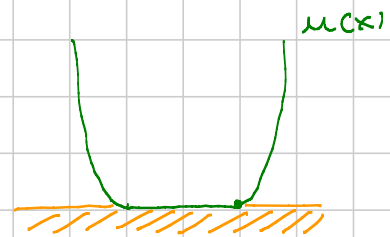
Remark To be rigorous, we need to prove the existence of $x(t)$ by means of the implicit function theorem, as in the point-to-curve problem. *Lecture*

$$\Phi(t, x) := u(x) + t v(x) - \varphi(x) = 0$$

Regularity of minimizer with obstacle $u(x) \geq \varphi(x)$

- If $\varphi \in C^1$, then $u \in C^1$
- If $\varphi \in C^{1,1}$ (φ is Lipschitz), then $u \in C^{1,1}$ (if L_φ is strictly increasing)
- If $\varphi \in C^\infty$, then $u \in C^{1,1}$

In the motivating example, the solution is "only" $C^{1,1}$ and $\varphi(x) \in C^\infty$.



u is discontinuous.

Calculus of Variations — LECTURE 41

Note Title

09/12/2016

LINEARIZATION EFFECTS "Small parameters"

Example 1 $\min \left\{ \int_0^1 (\dot{u}^2 + \dot{u}^4 + \varepsilon u + u^6) dx : u(0)=0, u(1)=0 \right\}$

For $\varepsilon > 0$ fixed existence of minimizers follows from the direct method. Moreover, the min $m_\varepsilon < 0$ because $u \equiv 0$ is a competitor, and it does not solve Euler eqn.

Idea: $m_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0^+$. Problem: find principal part.

Strategy: set $u = \varepsilon v$. The problem becomes

$$\varepsilon^2 \int_0^1 (\dot{v}^2 + \varepsilon^2 \dot{v}^4 + v + \varepsilon^4 v^6) dx$$

$$F_\varepsilon(u) \rightarrow F(u) = \int_0^1 (\dot{v}^2 + v) dx$$

It is reasonable to expect that $\min F_\varepsilon \rightarrow \min F(u)$ with DBC, and this can be easily computed.

This is true if we prove

- ① Γ -convergence
 - ② equicoerciveness of F_ε
- } $u \in C^0, L^1, L^2, L^{2^*}, \dots$

1.1 Gamma-liminf
then

$F_\varepsilon(u) \geq F(u)$. Therefore if $u_\varepsilon \rightarrow u$

$$\liminf_{\varepsilon \rightarrow 0^+} F_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon \rightarrow 0^+} F(u_\varepsilon) \geq F(u)$$

$\uparrow$ ineq.
 $\uparrow$ $F \text{ is LSC}$

①.2 Gamma-Limsup Trivial if $u \in C^1$ (we can take $u_\varepsilon \equiv u$) and C^1 is dense in L^2 w.r.t the energy $F(u)$.

② We have to prove that $F_\varepsilon(u_\varepsilon) \leq M$ implies existence of a converging subsequence.
Once again it follows from

$$F(u_\varepsilon) \leq F_\varepsilon(u_\varepsilon) \leq M$$

and usual compactness results for $F(u)$.

Example 2 $m_\varepsilon = \min \left\{ \int_0^1 \dot{u}^4 + \varepsilon u + u^6 : u(0) = u(1) = 0 \right\}$

Let us try $u = \varepsilon^\alpha v$

$$\begin{aligned} & \int_0^1 \varepsilon^{4\alpha} \dot{v}^4 + \varepsilon^{1+\alpha} v + \varepsilon^{6\alpha} v^6 \\ & \quad \quad \quad \uparrow \quad \quad \quad \text{choose } 4\alpha = 1 + \alpha \\ & \quad \quad \quad \Rightarrow \alpha = \frac{1}{3} \\ & = \varepsilon^{\frac{4}{3}} \int_0^1 \dot{v}^4 + v + \varepsilon^{\frac{2}{3}} v^6 \\ & \quad \quad \quad \underbrace{\hspace{10em}}_{F_\varepsilon(v)} \rightarrow F(v) := \int_0^1 \dot{v}^4 + v \end{aligned}$$

As before we have Gamma-convergence and equicoerciveness.

Example 3 $m_\varepsilon := \min \left\{ \int_0^1 (\dot{u}^2 + \dot{u}^8 + u^3 + u^6) dx : u(0) = 0, u(1) = \varepsilon \right\}$

Once again we set $u = \varepsilon v \rightsquigarrow v(0) = 0, v(1) = 1$

$$\begin{aligned} & \int_0^1 (\varepsilon^2 \dot{v}^2 + \varepsilon^8 \dot{v}^8 + \varepsilon^3 v^3 + \varepsilon^6 v^6) dx \\ & \quad \quad \quad \varepsilon^2 \int_0^1 (\dot{v}^2 + \varepsilon^6 \dot{v}^8 + \varepsilon v^3 + \varepsilon^4 v^6) dx \\ & \quad \quad \quad \underbrace{\hspace{10em}}_{F_\varepsilon(v)} \rightarrow F(v) := \int_0^1 \dot{v}^2 dx \end{aligned}$$

The minimum of $F(v)$ with BC is $\frac{1}{2}$ and therefore

$$m_\varepsilon \sim \frac{1}{2} \varepsilon^2.$$

Remark. We have also convergence of minimizers

$$v_\varepsilon(x) \rightarrow x \text{ uniformly.}$$

Again: we have to prove Gamma-conv. and equicoerciveness.

Example 4 $m_\varepsilon := \min \left\{ \int_0^1 \sinh(u^2) + \sin(u^2) : u(0) = u(1) = \varepsilon \right\}$

As before $u = \varepsilon v$

$$\int_0^1 \sinh(\varepsilon^2 \dot{v}^2) + \sin(\varepsilon^2 v^2) =$$

$$= \varepsilon^2 \int_0^1 \frac{\sinh(\varepsilon^2 \dot{v}^2)}{\varepsilon^2} + \frac{\sin(\varepsilon^2 v^2)}{\varepsilon^2}$$

$$F_\varepsilon(v) \rightarrow F(v) := \int_0^1 \dot{v}^2 + v^2$$

Again we can prove Gamma-conv. and equicoerciveness.

$$F_\varepsilon(v) \geq \int_0^1 (\dot{v}^2 - 1) dx$$

↑

$$\sinh x \geq x \text{ when } x \geq 0$$

$$\frac{\sinh x}{x} \geq -1 \text{ when } x \geq 0$$

This provides equicoerciveness, and the key step in the liminf inequality.

Limsup inequality is easy in C^1 , which is dense in energy.

Example 5

$$u_\varepsilon := \left\{ \int_0^1 \sin(u^2) + \sin R(u^2) : u(0) = u(1) = \varepsilon \right\}$$

In this case the min. does not exist for every $\varepsilon > 0$ and the infimum is actually always -1 , because it is the min. of the relaxation which is

$$\int_0^1 -1 + \sin R(u^2)$$

Example 6

$$\int_0^1 \dot{u}^2 + u^4 \quad u(0) = u(1) = \varepsilon$$

$$u = \varepsilon v \rightsquigarrow \int_0^1 \varepsilon^2 \dot{v}^2 + \varepsilon^4 v^4 \quad v(0) = v(1) = 1$$

$$= \varepsilon^2 \int_0^1 \dot{v}^2 + \varepsilon^2 v^4$$

$$F_\varepsilon(v) \rightarrow F(v) = \int_0^1 \dot{v}^2$$

↑
has min = 0.

$$\int_0^1 \varepsilon^2 \dot{v}^2 + \varepsilon^4 v^4 = \varepsilon^4 \int_0^1 \frac{\dot{v}^2}{\varepsilon^2} + v^4$$

$F_\varepsilon(v) \rightarrow F(v)$

where

$$F(v) := \begin{cases} \int_0^1 v^4 & \text{if } v \text{ is constant} \\ +\infty & \text{otherwise (in } L^2 \text{ or } C^0 \text{ or...)} \end{cases}$$

Easy : • limsup inequality : can take $u_\varepsilon := u$
• equicoerciveness

$$F_\varepsilon(v) \leq M \rightsquigarrow \|\dot{v}\|_{L^2} \leq \varepsilon^2 M$$

• liminf ineq : follows from previous estimate
(in the limit $\dot{v}_\varepsilon \rightarrow 0$ strongly, so that the limit is constant)

Calculus of Variations

LECTURE 42

Note Title

09/12/2016

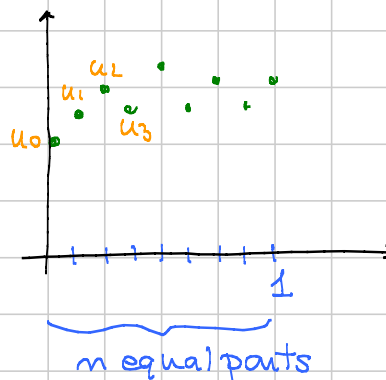
DISCRETE TO CONTINUUM MODELS

Functions $\rightsquigarrow$ values on a grid.

n = points in the grid

Function $\rightsquigarrow$ $(n+1)$ real numbers

$u_0, u_1, \dots, u_n$



$$F(u) = \int_0^1 \dot{u}^2 + u^2$$

$$F_n(u) = \frac{1}{n} \sum_{k=1}^n \left[\left(\frac{u_k - u_{k-1}}{\frac{1}{n}} \right)^2 + u_k^2 \right]$$

$$\int_0^1 u^2 \rightsquigarrow \frac{1}{n} \sum_{k=1}^n u_k^2$$

$$\int_0^1 \dot{u}^2 \rightsquigarrow \frac{1}{n} \sum_{k=1}^n \left(\frac{u_k - u_{k-1}}{\frac{1}{n}} \right)^2$$

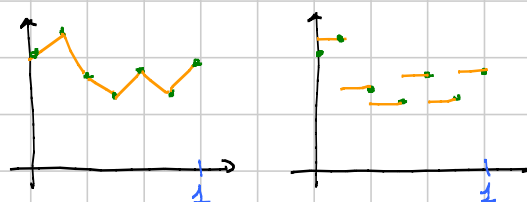
We note that F_n is defined in $\mathbb{R}^{n+1}$, while F is defined in L^2 or C^1 or C^0 or H^1 or...

We expect that $F_n \rightarrow F$ in the sense of Gamma-conv. This requires that F_n and F are defined in the SAME AMBIENT SPACE. We need to identify $\mathbb{R}^{n+1}$ with a suitable subset of L^2 (or C^0).

The are two standard ways to do this

1 - Piecewise affine extension

2 - Piecewise constant



Piecewise affine is good because gives rise to continuous functions and in that case

$$F_n(u) = F(u) \quad \text{as for the "derivative term"}$$

Piecewise constant is good because

$$F_n(u) = F(u) \quad \text{as for the "function term"}$$

and it is always possible, even when u take values in some strange metric space.

Result: $F_n \rightarrow F$ in C^0 or L^2 or ... in the sense of Gamma conv. + equicoerciveness.

Gamma-limit Take $u_n \rightarrow u_\infty$ in L^2

↑
for every n this is a
piecewise affine functions
on the grid with n samples

Need to prove that $\liminf F_n(u_n) \geq F(u_\infty)$

(Detail: $F_n(u) = +\infty$ when u is not piecewise affine or piecewise constant)

This is easy because $F_n(u_n) = F(u_n)$ if u_n is p.w.a. and F is LSC.

Equicoerciveness The same, because $F_n(u_n) \leq M$

$$\Rightarrow F(u_n) \leq M \Rightarrow \|u_n\|_{L^2} \leq M \quad \|u_n\|_{L^2} \leq M$$

Gamma limsup We need to prove that for every $u \in L^2$ there exists $u_n \rightarrow u$ s.t.

$$\limsup F_n(u_n) \leq F(u)$$

We can assume that $u \in H^1$ and therefore continuous.

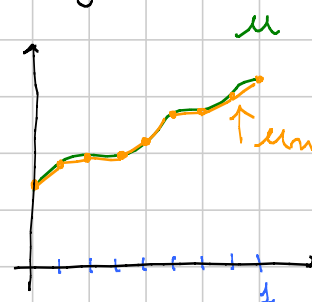
In this case u_n can be defined by connecting the dots !!!

In this case

$$u_n \rightarrow u \text{ unif.}$$

because u is unif. cont.

This guarantees the convergence of the part of $F_n(u_n)$ con to $\int u^2$



As for the other part, it is less that $\int u_n^2$ so again the limsup follows.

— o — o —

Euler equation in the discrete setting.

Notation : set $\varepsilon := \frac{1}{n}$ $D^\varepsilon u(x) = \frac{u(x+\varepsilon) - u(x)}{\varepsilon}$

If u is piecewise constant, then the sum of the derivative part is

$$F_\varepsilon(u) = \int_0^1 |D^\varepsilon u|^2 + u^2$$

Assume that u minimizes $F_\varepsilon(u)$ among all pwc functions

$$\begin{aligned} F_\varepsilon(u + tv) &= \int_0^1 |D^\varepsilon u|^2 + 2t D^\varepsilon u D^\varepsilon v + t^2 |D^\varepsilon v|^2 \\ &\quad + \int_0^1 u^2 + 2t uv + t^2 v^2 \end{aligned}$$

Derivative with $t=0$

$$2 \int_0^1 (D^\varepsilon u D^\varepsilon v + uv) = 0$$

Now we need "discrete integration by parts"

$$\int_0^1 D^\varepsilon f \cdot g = - \int_0^1 f D^{-\varepsilon} g$$

(if for example
 $f=0$ at the
"boundary")

↑
can be easily proved by shifting
indices in summations or
changing variables in integrals

Using this formula we obtain ($g := D^\varepsilon u$, $f := v$)

$$\int_0^1 (D^\varepsilon u D^\varepsilon v + uv) = \int_0^1 -D^{-\varepsilon} D^\varepsilon u \cdot v + uv$$

And therefore

$$\underline{D^{-\varepsilon} D^\varepsilon u} = u \quad \rightsquigarrow \quad \ddot{u} = u$$

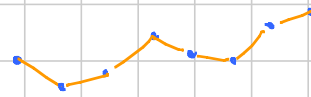
discrete counterpart
of the second derivative

$$D^{-\varepsilon} D^\varepsilon u = \frac{u(x+\varepsilon) + u(x-\varepsilon) - 2u(x)}{\varepsilon^2} \approx \ddot{u}(x)$$

$$\begin{pmatrix} -2 & 1 & & & \\ & 1 & -2 & 1 & 0 \\ & & 1 & -2 & 1 \\ 0 & & & 1 & -2 \end{pmatrix}$$

— 0 — 0 —

Second example Approximate the length of a curve by polygons



Given $n+1$ points in the plane or space $P_0, P_1, \dots, P_n$, define

$$F_n(P_0, P_1, \dots, P_n) := \sum_{k=1}^n |P_{k-1} P_k|$$

We would like to say that F converges to the length of a curve functional.

We have to say in which sense $(n+1)$ -uples of points converge to curves.

$(P_0, P_1, \dots, P_n) \rightsquigarrow$ p.w.a. curve defined in $[0, 1]$

So it makes sense to say that a family of such curves converges to some $\gamma: [0, 1] \rightarrow \mathbb{R}^2$ ($\mathbb{R}^3$)

Exercise Prove that $F_n(\gamma) \rightarrow \text{length of } \gamma$ w.r.t to C^0 topology.

— o — o —

Calculus of Variations - LECTURE 43

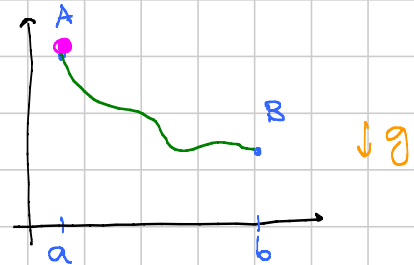
Note Title

14/12/2016

THE BRACHISTOCRONE PROBLEM

Shortest time

- 1- **MODEL** Finding the profile $u(x)$ in such a way that the time from A to B is as short as possible (ball is moving according to gravity)



$$(x(t), y(t)) \text{ trajectory} \quad y(t) = u(x(t)) \quad \dot{y}(t) = u'(x(t)) \dot{x}(t)$$

$$E(t) := \underbrace{\frac{1}{2} (\dot{x}^2 + \dot{y}^2)}_{\text{kinetic}} + gy = E(0) = gA$$

$\uparrow$
speed = 0 at $t=0$

$$\dot{x}^2 + \dot{u}^2 \dot{x}^2 = 2g(A - u(x(t)))$$

$$\dot{x}(t)^2 = 2g \frac{A - u(x(t))}{1 + u'^2(x(t))}$$

$$\dot{x}(t) = \sqrt{2g} \sqrt{\frac{A - u(x(t))}{1 + u'^2(x(t))}}$$

$$\int_0^T \sqrt{\frac{1 + u'^2(x(t))}{A - u(x(t))}} \dot{x}(t) dt = \int_0^T \sqrt{2g} dt = \sqrt{2g} T$$

$$y := x(t)$$

$$\int_a^b \sqrt{\frac{1 + u'^2(y)}{A - u(y)}} dy = \sqrt{2g} T$$

Minimizing T is equivalent to the min prob

$$\min \left\{ \int_a^b \sqrt{\frac{1 + u'^2}{A - u}} dx : u(a) = A, u(b) = B \right\}$$

Bad news: 1 - $L(s,p) := \frac{\sqrt{1+p^2}}{\sqrt{A-s}}$ has linear growth

2 - Denominator $\rightarrow 0$ when $x=a \leadsto$ improper integrals

We look for a minimizer $u \in C^0([a,b]) \cap C^1((a,b))$

2 - Euler equation Erdmann form $\tilde{u} L_p(u, \tilde{u}) = c + L(u, \tilde{u})$

$$\frac{\tilde{u}^2}{\sqrt{1+\tilde{u}^2}} \frac{1}{\sqrt{A-u}} = c + \frac{\sqrt{1+\tilde{u}^2}}{\sqrt{A-u}}$$

Brutal move ON: $\cancel{\tilde{u}^2} = c \sqrt{1+\tilde{u}^2} \sqrt{A-u} + 1 + \cancel{\tilde{u}^2}$

$$\sqrt{1+\tilde{u}^2} \sqrt{A-u} = c \leadsto 1+\tilde{u}^2 = \frac{c^2}{A-u}$$

$$\leadsto \tilde{u} = \pm \sqrt{\frac{c^2 - (A-u)}{A-u}}$$

ODE solvable by separation of variables

$$\int \sqrt{\frac{A-u}{c^2 - (A-u)}} du = \pm \int dx = \pm x + k$$

Variable change: $A-u = c^2 \sin^2 z \leadsto -du = 2c^2 \sin z \cos z$

$$\pm \int \frac{\cancel{c} \sin z}{\cancel{c} \cos z} 2c^2 \sin z \cancel{\cos z} dz = \pm x + k$$

$$= \pm 2c^2 \int \sin^2 z dz = \pm 2c^2 \left(\frac{1}{2} 2z - \frac{1}{4} \sin(2z) \right)$$

$$= \pm \frac{c^2}{2} (2z - \sin(2z))$$

One more variable change

$$x = \frac{c^2}{2} (\theta - \sin \theta) + k$$

$$u = A - c^2 \sin^2 z = A - \frac{c^2}{2} (1 - \cos \theta)$$

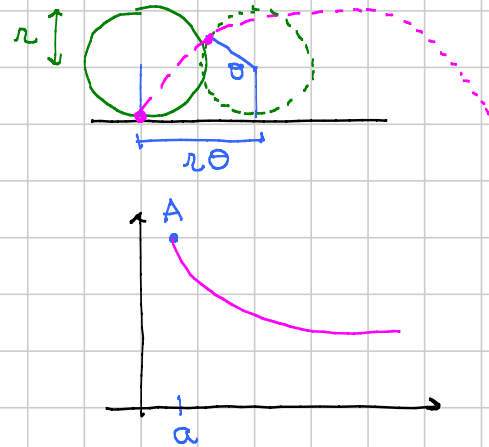
Setting $r := \frac{c^2}{2}$ we found that (choose $k=a$)

$$\begin{cases} x = r(\theta - \sin \theta) + a \\ y = A - r(1 - \cos \theta) \end{cases} \quad \text{Parametrization of a cycloid}$$

Standard form

$$x = r(\theta - \sin \theta) \quad \theta \in [0, 2\pi]$$

$$y = r(1 - \cos \theta)$$



Remark $f(\theta) := \theta - \sin \theta$ is

an invertible function (strictly increasing) which admits an inverse $g: \mathbb{R} \rightarrow \mathbb{R}$. From the first eq. we obtain

$$\theta = g\left(\frac{x-a}{r}\right)$$

3- System Does there exist $r > 0$ such that the cycloid passes through (b, B) ?

$$b = r(\theta - \sin \theta) + a$$

$$r(\theta - \sin \theta) = b - a > 0$$

$$B = A - r(1 - \cos \theta)$$

$$r(1 - \cos \theta) = A - B > 0$$

Dividing by r we find

$$\frac{1 - \cos \theta}{\theta - \sin \theta} = \frac{A - B}{b - a} > 0$$

$\psi(\theta)$

Claim: $\psi: (0, 2\pi) \rightarrow (0, +\infty)$ is invertible.

If claim is true, then we can find a unique $\theta \in (0, 2\pi)$ and a unique r that solve the system.

Nice exercise: $\psi'(\theta) < 0 \quad \forall \theta \in (0, 2\pi)$



Theorem 1 Let $b > a$ and let $A > B$. Let (r_0, θ_0) be the solution to the system.

Let us set

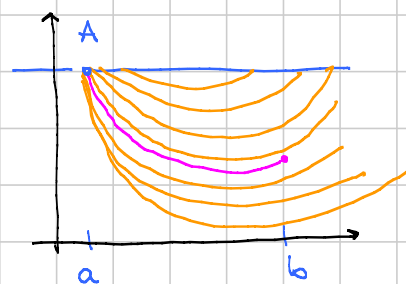
$$u_0(x) := A - r_0 \left(1 - \cos \left(\theta_0 \left(\frac{x-a}{r_0} \right) \right) \right) \quad \forall x \in [a, b]$$

Then $u_0 \in C^0([a, b]) \cap C^1((a, b))$ is a solution to Euler equ. of the original pbm and satisfies the DBC.

Theorem 2 The function $u_0(x)$ defined above is a GM for the original pbm

Proof 1 Use calibrations $\leadsto$ describing a Weierstrass field (Weierstrass excess is ≥ 0 because $L(x, s, p)$ is convex wrt p)

All the cycloids as $r > 0$ enclose the region with $x > a$ and $u < A$



Any other solution in this region satisfies WRF and therefore it is worse than u_0 .

Proof 2 Convexity trick. Let us set $A - u = \frac{1}{2} v^2$ so that

$$-\ddot{u} = r\ddot{v} \quad \sqrt{\frac{1+\dot{u}^2}{A-u}} = \sqrt{\frac{1+v^2\dot{v}^2}{v^2}} \sqrt{2} = \sqrt{2} \sqrt{\dot{v}^2 + \frac{1}{v^2}}$$

and therefore in the new variable

$$\min \left\{ \int_a^b \sqrt{\frac{1+\dot{u}^2}{A-u}} dx : BC \right\} = \sqrt{2} \left\{ \int_a^b \sqrt{\dot{v}^2 + \frac{1}{v^2}} dx : \text{modified } BC \right\}$$

$v(a)=0, v(b)=\dots$

and $v_0(x)$ corresponding to $u_0(x)$ is a solution to the new Euler equation.

Surprise $L(s, p) := \sqrt{p^2 + \frac{1}{s^2}}$ is convex wrt (p, s) . (compute the Hessian matrix)

Calculus of Variations — LECTURE 44

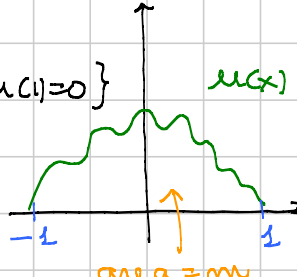
Note Title

14/12/2016

CARTESIAN DIDO'S PROBLEM

Max area with given perimeter or, which is the same, minimizing the perimeter given the area.

Cartesian version

$$\min \left\{ \underbrace{\int_{-1}^1 \sqrt{1+\dot{u}^2} dx}_{\text{length of graph}} : \underbrace{\int_{-1}^1 u dx = m}_{\text{given area}}, u(-1)=u(1)=0 \right\}$$


$$F(u) := \int_{-1}^1 \sqrt{1+\dot{u}^2} dx \quad G(u) = \int_{-1}^1 u dx$$

Bad news: linear growth w.r.t $\dot{u}$.

Lagrange multipliers: if u is a min point, then

→ either $\delta G(u, v) = 0$ for every v admissible

→ or there exists $\lambda \in \mathbb{R}$ such that $\delta(F + \lambda G) = 0$

First variation of G : $(L_p)' = L_s$
 $0 = 1$ no solution

First variation of $F + \lambda G$: $L(x, s, p) = \sqrt{1+p^2} + \lambda s$

$$(L_p)' = L_s \quad \left(\frac{\dot{u}}{\sqrt{1+\dot{u}^2}} \right)' = \lambda \quad ; \quad \frac{\dot{u}}{\sqrt{1+\dot{u}^2}} = \lambda x + k$$

$$\frac{\ddot{u}^2}{1+\dot{u}^2} = (\lambda x + k)^2 ; \quad \ddot{u}^2 = (\lambda x + k)^2 + (\lambda x + k)^2 \dot{u}^2$$

$$\dot{u}^2 = \frac{(\lambda x + k)^2}{1 - (\lambda x + k)^2}$$

$$\dot{u} = \pm \frac{\lambda x + k}{\sqrt{1 - (\lambda x + k)^2}}$$

$$= \left(\pm \sqrt{1 - (\lambda x + k)^2} \right)' \frac{1}{\lambda}$$

Finally we obtain that

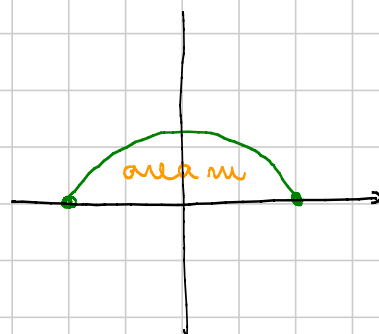
$$\lambda u = \pm \sqrt{1 - (\lambda x + k)^2} + c \rightsquigarrow (\lambda u - c)^2 = 1 - (\lambda x + k)^2$$

$$(\lambda x + c)^2 + (\lambda x + k)^2 = 1$$

circle of radius $\frac{1}{\lambda}$ depending on three parameters λ, c, k

we have three equations to fulfil:

$$\left\{ \begin{array}{l} u(-1) = 0 \\ u(1) = 0 \\ \int_{-1}^1 u(x) dx = m \end{array} \right\} \rightarrow c = 0 \text{ (center is on y axis)}$$



Not difficult to see that the solution is unique, if it exists.

However: the solution exists if and only if $-\frac{\pi}{2} \leq m \leq \frac{\pi}{2}$

Theorem 1 For every $m \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ the Euler equation admits (at least) one solution $u_0(x)$ given by an arc of a circle with radius $\frac{1}{\lambda}$.

If m is not in the range, then Euler equ. has no solution

Questions : $\rightarrow$ is $u_0(x)$ a GM?

$\rightarrow$ what happens if $m > \frac{\pi}{2}$.

sufficient condition

LEMMA (Trivial lemma for constrained minimization)

In a quite general setting, assume that we need to solve

$$\min \{ F(u) : G(u) = m \}$$

Assume that (u_0, λ_0) is a solution to option 2 of Lagrange multipliers, namely u_0 is a stationary point for $F(u) + \lambda_0 G(u)$

Assume that u_0 is a minimum point for $F(u) + \lambda_0 G(u)$.
Then u_0 is a min point for the original problem.

Proof.

$$F(u) = F(u) + \lambda_0 G(u) - \lambda_0 G(u) \geq F(u_0) + \lambda_0 G(u_0) - \lambda_0 G(u)$$

$\uparrow$
 u_0 is min point
for $F + \lambda G$

$\xrightarrow{\text{equal}}$
 if $G(u_0) = G(u)$
 $= m$

This proves that $F(u) \geq F(u_0)$ if $G(u) = m$.

In our case it is enough to check that $u_0(x)$ is a minimum point for

$$(F + \lambda G)u = \int_{-1}^1 \sqrt{1+u^2} + \lambda u$$

Proof 1 Again by convexity of $L(x, s, p) := \sqrt{1+p^2} + \lambda s$
(the sign of λ does not matter)
wrt the pair (p, s)

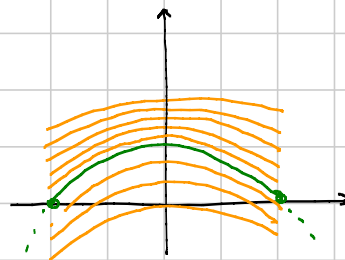
Proof 2 Weierstrass excess is ≥ 0 due to convexity in p .
It is enough to find a Weierstrass field

To this end, it is enough to consider vertical translation of $u_0(x)$.

This is true because, if u_0 solves

$$\left(\frac{\dot{u}}{\sqrt{1+\dot{u}^2}} \right)' = \lambda$$

then $u_0(x) + c$ is again a solution.

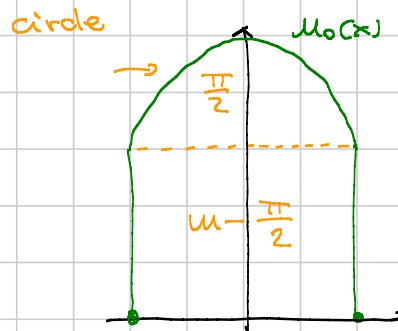


What happens if $m > \frac{\pi}{2}$?

The min. does not exist (Euler equ. has no solution) and the supremum is represented by a half stadium

The "minimizer" is discontinuous at the boundary.

The price of vertical jump of height J is exactly J



$$\int_0^{\frac{1}{m}} \sqrt{1+\dot{u}_m^2} = \frac{1}{m} \sqrt{1+m^2} \rightarrow 1$$

$$\int_0^{\frac{1}{m}} \sqrt{1+\dot{u}_m^2} = \frac{1}{m} \sqrt{1+J^2 m^2} \rightarrow J$$

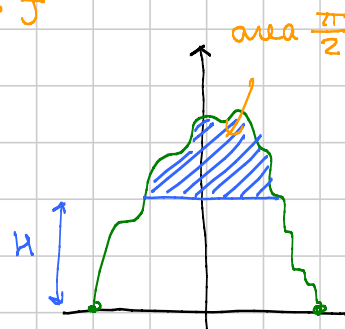
Sketch of proof Take a level H such that "area above H " is $\frac{\pi}{2}$. Then

- the perimeter above H is at least π
- H can be estimated because

$$2H = \text{area rectangle} \geq m - \frac{\pi}{2}$$

- Perimeter below $H \geq 2H$.

Total perimeter $\geq 2H + \pi \geq m - \frac{\pi}{2} + \pi = \text{perimeter of } \frac{1}{2} \text{ stadium}$



Calculus of Variations — LECTURE 45

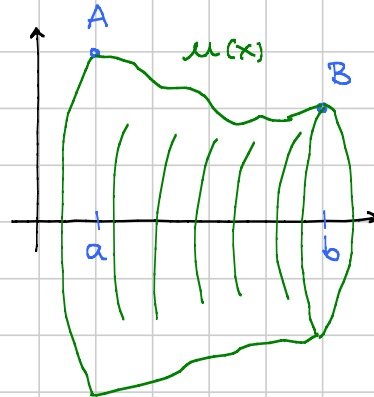
Note Title

16/12/2016

MINIMAL SURFACE OF REVOLUTION $[a, b] \subseteq \mathbb{R}$ interval A, B positive real numbers

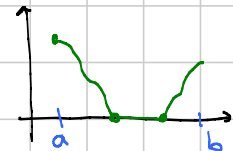
Finding $u(x)$ in such a way that the area of the revolution surface is min

$$\min \left\{ \underbrace{2\pi \int_a^b u \sqrt{1+u'^2} dx}_{\text{Area}} : u(a) = A, u(b) = B \right\}$$



Bad news: $L(s, p) := s \sqrt{1+p^2}$ has linear growth wrt p .

Remark We also assume that $u(x) \geq 0$ in $[a, b]$. This follows from minimality and usual truncation argument.

Euler equation Erdmann form

$$u' L_p(u, u') = c + L(u, u')$$

$$u' \frac{u' u}{\sqrt{1+u'^2}} = c + u \sqrt{1+u'^2}$$

$$u \cancel{u'^2} = c \sqrt{1+u'^2} + u + u \cancel{u'^2}$$

$$\sqrt{1+u'^2} = \frac{u}{c}$$

$$u'^2 = \frac{u^2}{c^2} - 1$$

$$u' = \pm \sqrt{\frac{u^2}{c^2} - 1}$$

Brutal mode! $\int \frac{c du}{\sqrt{u^2 - c^2}} = \int \pm dx = \pm x + k$

$u = cv \leadsto du = c dv$ " $\int \frac{c \cdot c dv}{c \sqrt{v^2 - 1}} = c \arccos v = \pm x + k$

Therefore $\arccos\left(\frac{u}{c}\right) = \frac{x}{c} + k$ so that finally

$$u(x) = c \cos\left(\frac{x}{c} + k\right) \quad \text{with } c > 0, k \in \mathbb{R}$$

System (Courtesy Paolo Tilli) Finding c and k in order to fulfil BCs.

Let us assume wlog that $a = -r$, $b = r$ for some $r > 0$

$$u(-r) = c \cos\left(-\frac{r}{c} + k\right) = A$$

$$u(r) = c \cos\left(\frac{r}{c} + k\right) = B$$

let us $t := \frac{r}{c}$ $C = \cos$ $S = \sin$ $T = \tan$

$$\cos\left(-\frac{r}{c} + k\right) = \frac{A}{c} = \frac{A}{r} t$$

$$\cos\left(-\frac{r}{c} + k\right) = \frac{B}{c} = \frac{B}{r} t$$

so that $C(-t + k) = \frac{A}{r} t$ $C(t + k) = \frac{B}{r} t$

$$C(t)C(k) - S(t)S(k) = \frac{A}{r} t$$

$$C(t)C(k) + S(t)S(k) = \frac{B}{r} t$$

$$C(t)C(k) = \frac{B+A}{2r} t$$

$$S(t)S(k) = \frac{B-A}{2r} t$$

$$C(k) = \frac{B+A}{2r} \frac{t}{C(t)}$$

$$S(k) = \frac{B-A}{2r} \frac{t}{S(t)}$$

$$C(k)^2 - S(k)^2 = \left(\frac{B+A}{2r}\right)^2 \frac{t^2}{C(t)^2} - \left(\frac{B-A}{2r}\right)^2 \frac{t^2}{S(t)^2} = 1$$

$$1 + \left(\frac{B-A}{2r} \right)^2 \frac{t^2}{S(t)^2} = \left(\frac{B+A}{2r} \right)^2 \frac{t^2}{C(t)^2}$$

and finally

$$\boxed{\frac{C(t)^2}{t^2}} + \left(\frac{B-A}{2r} \right)^2 \boxed{\frac{1}{T(t)^2}} = \left(\frac{B+A}{2r} \right)^2$$

↑ convex ↑ convex

Want: positive solutions of this equ.

Standard check: $\frac{C(t)}{t}$, $\frac{1}{T(t)}$ are convex nonnegative functions for $t > 0$

and the square is again convex.

Therefore we have 0/1/2 solutions depending on the RHS

Special case $A=B \rightsquigarrow K=0$

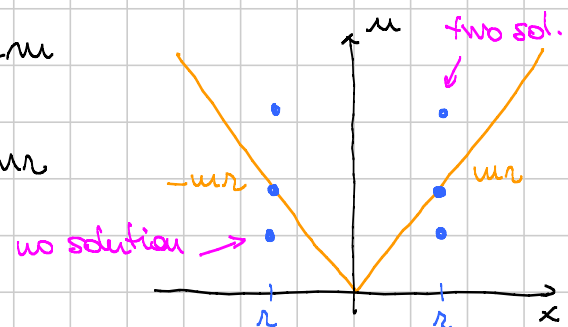
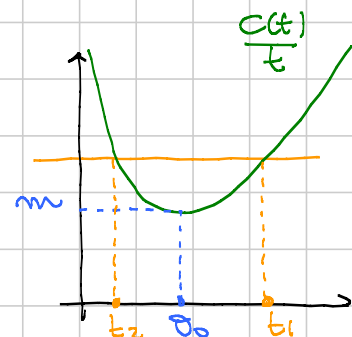
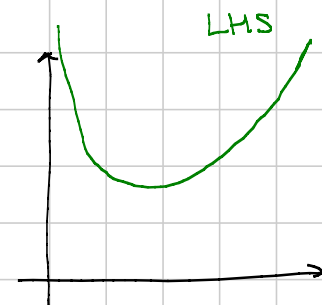
$$\frac{C(t)^2}{t^2} = \left(\frac{A}{r} \right)^2$$

$$\boxed{\frac{C(t)}{t} = \frac{A}{r}}$$

• No solution if $\frac{A}{r} < m$; $A < mr$

• One solution if $A = mr$

• Two solutions if $A > mr$

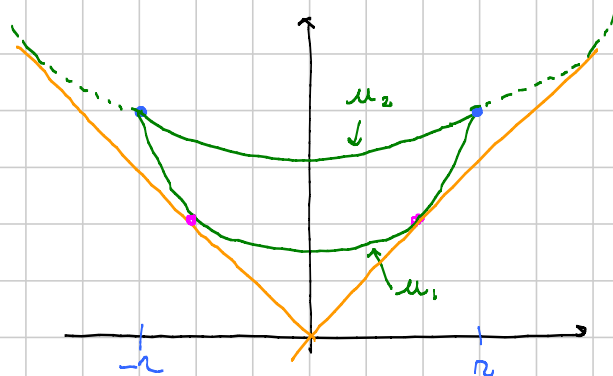


Case with two solutions

$$u_1(x) = c_1 \operatorname{cosec} \left(\frac{x}{c_1} \right) \quad c_1 < c_2$$

$$u_2(x) = c_2 \operatorname{cosec} \left(\frac{x}{c_2} \right)$$

Both solutions are tangent to $|x|$, but u_1 is the solution of $[-1, r]$ and u_2 outside



For $x > 0$: $c \operatorname{cosec} \left(\frac{x}{c} \right) \geq mx$; $\frac{\operatorname{cosec} \left(\frac{x}{c} \right)}{\frac{x}{c}} \geq m$

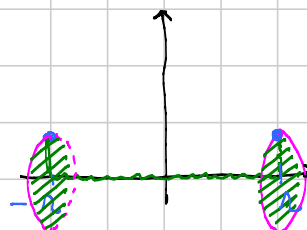
with equality $\Leftrightarrow \frac{x}{c} = \theta_0 \Leftrightarrow x = c\theta_0$

We have to understand when $c\theta_0 > r \Leftrightarrow \frac{r}{c} < \theta_0$

Observe that t_1 and t_2 are on opposite sides of θ_0 .

FACTS

- ① When there are 2 solutions ($A > mr$) then
 $u_2(x)$ is SLM
 $u_1(x)$ is not even WLM
- ② If $A > mr$ and r is small enough, then $u_2(x)$ is GM
- ③ If $A < mr$, then the min. prob. has no solution and the inf. is achieved by a discontinuous solution $\equiv 0$
 (in some sense the solutions are the two disks).

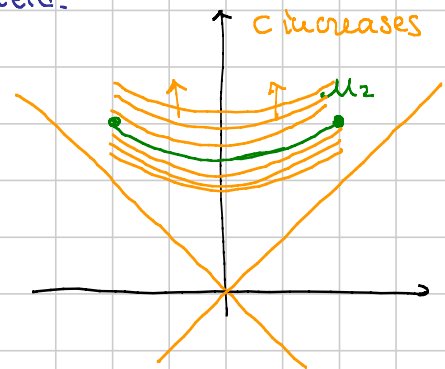


Proof That $u_2(x)$ is SLM $L(s,p)$ is convex in p
 $\Rightarrow$ excess ≥ 0 .

Enough to find a Weierstrass field.

$$u_2(x) = c \cosh\left(\frac{x}{c}\right)$$

As c varies this is a field.



Proof that u_2 is GM if ϵ is small

It is enough to prove that it is not convenient to exit the rectangle.

The exit cost can be estimated by

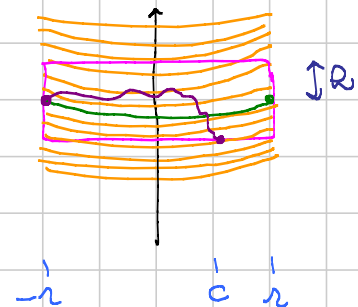
$$\int_a^b \dots \geq \int_a^c u \sqrt{1 + \tilde{u}^2} dx \geq \int u \tilde{u}$$

$$= \left[\frac{1}{2} u^2 \right]_a^c$$

$$= \frac{1}{2} \text{ something depending on } \epsilon$$

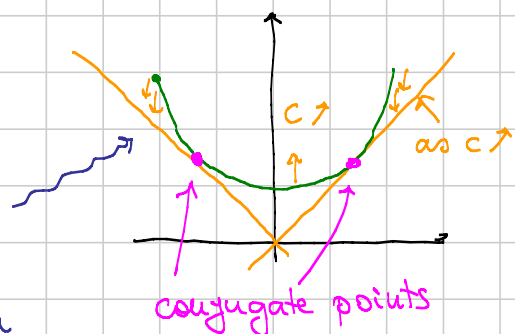
$$> F(u_2)$$

$\uparrow$
true if ϵ is small enough



Sketch of why u_1 is not DLM

this is equivalent
to saying that
sol. to Jacobi equ
do not have
constant sign.



Calculus of Variations

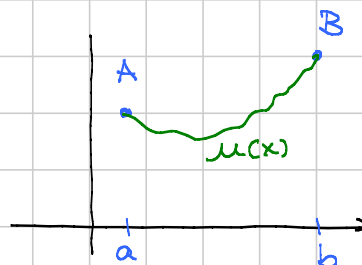
LECTURE 46

Note Title

16/12/2016

HEAVY CHAIN PROBLEM

Finding the shape of a chain or string subject to its own weight



UNEXTENDIBLE = FIXED LENGTH

Given (a, A) and (b, B) and $\ell > 0$

$$\min \left\{ \int_a^b u \sqrt{1+u'^2} dx : \int_a^b \sqrt{1+u'^2} dx = \ell \right\}$$

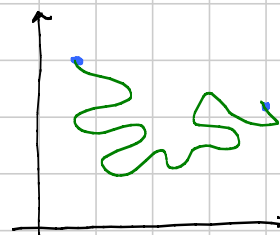
integral of u along the curve
= proportional to vertical coordinate of barycenter of the curve

Above is the cartesian formulation. More generally we could consider the parametric approach

$$\gamma: [0, 1] \rightarrow \mathbb{R}^2 \quad \gamma(t) = (x(t), y(t))$$

$$\min \left\{ \int_0^1 y \sqrt{\dot{x}^2 + \dot{y}^2} dt : \int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2} dt = \ell \right\}$$

$$\text{BC: } \begin{array}{ll} x(0) = a & y(0) = A \\ x(1) = b & y(1) = B \end{array}$$



Euler eqn with Lagrange multipliers
(cartesian formulation)

$$F(u) := \int_a^b u \sqrt{1+u'^2} dx$$

$$G(u) := \int_a^b \sqrt{1+u'^2} dx$$

→ Case $\delta G = 0$

→ Case $\delta (F + \lambda G) = 0$ for some $\lambda \in \mathbb{R}$

$$\boxed{\delta G = 0} \rightsquigarrow \left(\frac{\dot{u}}{\sqrt{1+\dot{u}^2}} \right)' = 0 \rightsquigarrow \underbrace{\frac{\dot{u}}{\sqrt{1+\dot{u}^2}} = c}_{\text{injective}} \rightsquigarrow \dot{u} = c_1$$

$\rightsquigarrow u$ is a line. $\rightsquigarrow \ell = |AB| \rightsquigarrow$ the line is actually the only competitor

$$\boxed{\delta (F + \lambda G) = 0} \quad L(s, p) := 5\sqrt{1+p^2} + \lambda\sqrt{1+p^2} \\ = (5+\lambda)\sqrt{1+p^2}$$

Up to λ , the Lagrangian is the same as in the min. surface of revolution. This implies that

$u(x) - \lambda$ is a stationary point for min surf. rev.



$u(x)$ is a stationary point for the heavy chain.

It follows that our solutions are of the form

$$u(x) = c \cosh\left(\frac{x}{c} + k\right) + \lambda$$

depending on c, k, λ (and we have 2 BCs + length = ℓ)

System

$$c \cosh\left(\frac{a}{c} + k\right) + \lambda = A$$

$$c \cosh\left(\frac{b}{c} + k\right) + \lambda = B$$

$$\boxed{\ell =} \int_a^b \sqrt{1+\dot{u}^2} dx = \int_a^b \sqrt{1+\sinh^2\left(\frac{x}{c} + k\right)} dx \\ = \int_a^b \cosh\left(\frac{x}{c} + k\right) dx = \left[c \sinh\left(\frac{x}{c} + k\right) \right]_a^b$$

Assume now that $[a, b] = [-1, 1]$ and set $\frac{\ell}{c} =: t$.

$$c C(-t+k) - c C(t+k) = A - B$$

$$c S(t+k) - c S(-t+k) = \ell$$

$$C(t)C(k) - S(t)S(k) - C(t)C(k) - S(t)S(k) = \frac{A-B}{c}$$

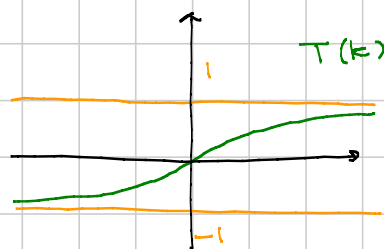
$$S(t)C(k) + C(t)S(k) + S(t)C(k) - C(t)S(k) = \frac{\ell}{c}$$

$$S(t)S(k) = \frac{B-A}{2c}$$

$$S(t)C(k) = \frac{\ell}{2c}$$

Divide: $T(k) = \frac{B-A}{\ell} \in [-1, 1]$

True 😊
because of geometry



so we can find a unique k

$$S(t) = \frac{\ell}{2c} \frac{1}{C(k)} = \frac{\ell}{2c} \sqrt{1 - T^2(k)}$$

$$= \frac{\ell}{2c} \sqrt{1 - \left(\frac{B-A}{\ell}\right)^2}$$

$$C^2 - S^2 = 1$$

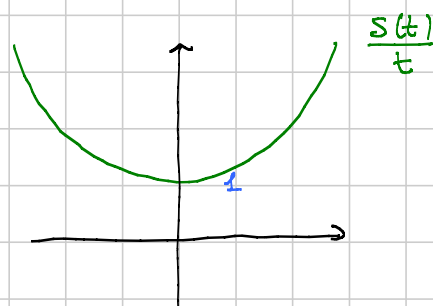
$$1 - T^2 = \frac{1}{C^2}$$

$$S(t)^2 = \left(\frac{\ell}{2c}\right)^2 \left(1 - \left(\frac{B-A}{\ell}\right)^2\right)$$

$$= \left(\frac{\ell}{2\ell}\right)^2 t^2 \left(1 - \left(\frac{B-A}{\ell}\right)^2\right)$$

$$\left[\frac{S(t)}{t}\right]^2 = \underbrace{\left(\frac{\ell}{2\ell}\right)^2 \left(1 - \left(\frac{B-A}{\ell}\right)^2\right)}_{\geq 1}$$

$$\frac{\ell}{2c} = \frac{\ell t}{2c\ell} = \frac{\ell t}{2\ell}$$



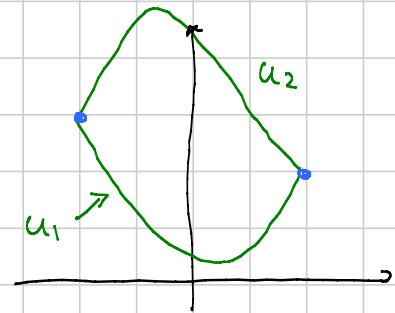
True $1 - \left(\frac{B-A}{\ell}\right)^2 \geq \left(\frac{2\ell}{\ell}\right)^2 \Leftrightarrow \underbrace{(2\ell)^2 + (B-A)^2}_{\text{distance AB}} \leq \ell^2$

Since conditions are satisfied, the system can be solved.
Actually there are two solutions, one with $c > 0$ and one with $c < 0$.

They are "symmetric":

$u_1(x)$ is the GM $\rightarrow$ not proved

$u_2(x)$ is the global maximum



The result is proved if we can show that the min exist.

How to prove it?

Assume we want $\min \left\{ \int_a^b u (1+u^2) dx : \int u^2 = L \right\}$

Can we apply direct method? More or less

$\rightarrow$ the constraint $\int u^2 = L$ gives compactness (+ BC)

$\rightarrow$ when we pass $\int_a^b u_n^2 = L$ to the limit we just obtain that $\int_a^b u_\infty^2 \leq L$.

Fortunately

$$\min \left\{ \int \dots : \int u^2 = L \right\} = \min \left\{ \int \dots : \int u^2 \leq L \right\}$$

stable by weak convergence

In this case we say that the min SATURATES the constraint.

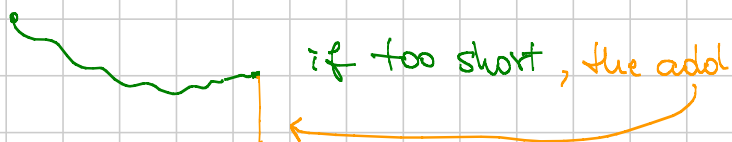
Alternative approach : reparam. the curves

$$\gamma: [0, \ell] \rightarrow \mathbb{R}^2 \quad \dot{x}^2 + \dot{y}^2 = 1$$

$$\min \left\{ \int_0^\ell y \sqrt{\dot{x}^2 + \dot{y}^2} : \dot{x}^2 + \dot{y}^2 = 1 \right\} = \min \left\{ \int_0^\ell y \sqrt{\dot{x}^2 + \dot{y}^2} : \dot{x}^2 + \dot{y}^2 \leq 1 \right\}$$

stable by
weak convergence

The last prob. has a solution (by direct method)
and the solution saturates the condition.



Calculus of Variations

LECTURE 47

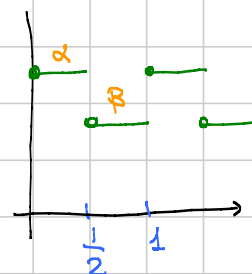
Note Title

21/12/2016

HOMOGENIZATION PROBLEMSGreek function

$$c: [0, 1] \rightarrow \mathbb{R}$$

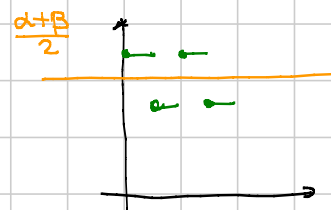
$$c(x) = \begin{cases} \alpha & \text{if } x \in [0, \frac{1}{2}) \\ \beta & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$



Let us extend to $c: \mathbb{R} \rightarrow \mathbb{R}$ by periodicity.

Let us set $c_n(x) := c(nx)$ so that c_n is $\frac{1}{n}$ -periodic

Exercise $c_n \rightarrow \frac{\alpha+\beta}{2}$ in $L^p(a,b)$
for every $p \geq 1$ ($p \neq +\infty$) and
for every $(a,b) \subseteq \mathbb{R}$.



Functionals: $G_n(u) := \int_0^1 \dot{u}^2 + c_n(x) u^2$ $\leadsto$ easy case
 $F_n(u) := \int_0^1 c_n(x) \dot{u}^2 + u^2$ $\leadsto$ diff. case

Theorem 1 $G_n(u) \xrightarrow{\Gamma} G_\infty(u)$, where

$$G(u) = \int_0^1 \dot{u}^2 + \frac{\alpha+\beta}{2} u^2$$

More generally, if $G_n(u) = \int_0^1 \dot{u}^2 + \underbrace{f_n(x)}_{\geq 0} u^2$

and $f_n \rightarrow f_\infty$ weakly, then

$$G_n(u) \rightarrow \int_0^1 \dot{u}^2 + f_\infty(x) u^2$$

Proof. **Liminf ineq.** Let $u_n \rightarrow u_\infty$ in L^2 (or C^0) and let us assume that

$$\liminf G_n(u_n) < +\infty$$

and actually a limit. Since $f_n(x) \geq 0$, we have a bound on $\|u_n\|_{L^2}$ and therefore (up to subsequences)

$$u_n \rightarrow u_\infty \quad u_n \rightarrow u_\infty \text{ unif.}$$

We claim that $\liminf_{n \rightarrow \infty} \int_0^1 u_n^2 \geq \int_0^1 u_\infty^2$ (classical)

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) u_n^2 = \int_0^1 f_\infty(x) u_\infty^2.$$

$$f_n \rightarrow f_\infty, \quad u_n \rightarrow u_\infty \text{ unif.} \Rightarrow u_n^2 \rightarrow u_\infty^2 \text{ unif.}$$

$$\Rightarrow u_n^2 \rightarrow u_\infty^2 \text{ in } L^2 \text{ strong.}$$

$\int_0^1 f_n \cdot u_n^2$ is the scalar prod. of a strongly conv. and a weakly conv. sequence.

Limsup ineq. For every $u \in L^2$ we used a recovery sequence. It is enough if $G(u) < +\infty$, and therefore $u \in H^1$, and therefore $u \in C^0$.

Now we can use $u_n \equiv u$, and trivially

$$\int_0^1 f_n(x) u^2 \rightarrow \int_0^1 f_\infty(x) u^2.$$

Theorem 2 Assume that $\alpha > 0$, $\beta > 0$. Then

$$F_n(u) \xrightarrow{\Gamma} F_\infty(u) = \int_0^1 \frac{2\alpha\beta}{\alpha+\beta} \dot{u}^2 + u^2$$

↑
Harmonic mean of α and β

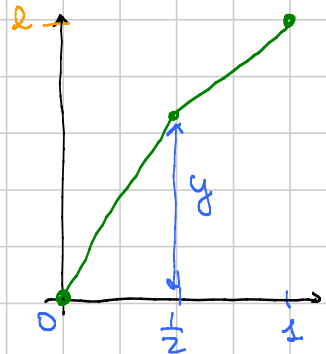
[It is enough to consider $F_n(u) := \int_0^1 c_n(x) \dot{u}^2$]

CELL PROBLEM

$$\min \left\{ \int_0^1 c(x) \dot{u}^2 dx : u(0)=0, u(1)=l \right\}$$

$$\int_0^{\frac{1}{2}} \alpha \dot{u}^2 + \int_{\frac{1}{2}}^1 \beta \dot{u}^2$$

We can restrict to functions as here $\rightarrow$



$$\frac{1}{2} \alpha (2y)^2 + \frac{1}{2} \beta (2(l-y))^2$$

$$= 2 [\alpha y^2 + \beta (l-y)^2] \rightsquigarrow \min \text{ wrt } y$$

$$\alpha y - \beta (l-y) = 0 \Leftrightarrow y(\alpha+\beta) = \beta l \Leftrightarrow y = \frac{\beta}{\alpha+\beta} l$$

Replacing this value, the min is

$$2 \left[\alpha \frac{\beta^2}{(\alpha+\beta)^2} l^2 + \beta \frac{\alpha^2}{(\alpha+\beta)^2} l^2 \right] = 2 \frac{\alpha\beta}{(\alpha+\beta)^2} l^2 (\alpha+\beta)$$

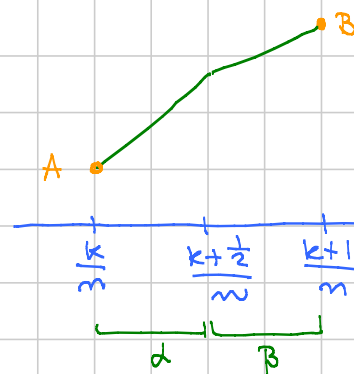
$$= \frac{2\alpha\beta}{\alpha+\beta} l^2$$

Conclusion The minimum of the cell problem is the same as

$$\min \left\{ \int_0^1 \frac{2\alpha\beta}{\alpha+\beta} \dot{u}^2 dx : u(0)=0, u(1)=l \right\}$$

Modified cell pbm

$$\min \left\{ \int_{\frac{k}{n}}^{\frac{k+1}{n}} C_n(x) \dot{u}^2 : u\left(\frac{k}{n}\right) = A, u\left(\frac{k+1}{n}\right) = B \right\}$$



As before the min is

$$\frac{2\alpha\beta}{\alpha+\beta} (B-A)^2 n$$

$$[n(B-A)]^2 \cdot \frac{1}{n}$$

$\uparrow$ derivative of the line $\uparrow$ interval of integration

Special case let us assume that $u_n \rightarrow u_\infty$ unif. where $u_\infty(x) = 2x$ on $[0, 1]$.

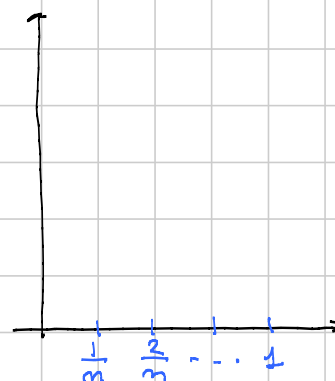
Then

$$\liminf_{n \rightarrow \infty} F_n(u_n) \geq F_\infty(u_\infty)$$

Proof.

$$F_n(u_n) = \int_0^1 C_n(x) \dot{u}_n^2 dx$$

$$= \sum_{k=1}^n \underbrace{\int_{\frac{k-1}{n}}^{\frac{k}{n}} C_n(x) \dot{u}_n^2 dx}_{\text{modified cell pbm}}$$



$$\geq \sum_{k=1}^n \left(\frac{2\alpha\beta}{\alpha+\beta} \right) n \left[u_n\left(\frac{k}{n}\right) - u_n\left(\frac{k-1}{n}\right) \right]^2 \quad n \sum \alpha_k^2 \geq (\sum \alpha_k)^2$$

$$\begin{aligned} &\stackrel{\text{C.S.}}{\geq} \frac{2\alpha\beta}{\alpha+\beta} \left(\sum_{k=1}^n u_n\left(\frac{k}{n}\right) - u_n\left(\frac{k-1}{n}\right) \right)^2 \\ &= \frac{2\alpha\beta}{\alpha+\beta} [u_n(1) - u_n(0)]^2 = \frac{2\alpha\beta}{\alpha+\beta} 2^2 = F_\infty(u_\infty) \end{aligned}$$

Analogous case Let $(a,b) \subseteq \mathbb{R}$ be any interval.
Assume that

$$u_n \rightarrow u_\infty(x) \text{ unif. in } [a,b].$$

$$\text{Then } \liminf_{n \rightarrow \infty} F_n(u_n) \geq \frac{2\alpha\beta}{\alpha+\beta} [u_\infty(b) - u_\infty(a)]^2 \frac{1}{b-a}$$

If $u_\infty(x)$ is a line, then the RHS is actually $F_\infty(u_\infty)$

Proof : more or less the same as above.
— o — o —

Proof that $F_n \xrightarrow{\Gamma} F_\infty$ always

Limiting inequality Let us fix $k \geq 1$. Let $u_n \rightarrow u_\infty$ in L^2
with $\liminf_{n \rightarrow \infty} F_n(u_n) < \infty$.

As always we obtain that $u_n \rightarrow u_\infty$ unif.

Let v_k be the piecewise approx of u_∞ with k subintervals.

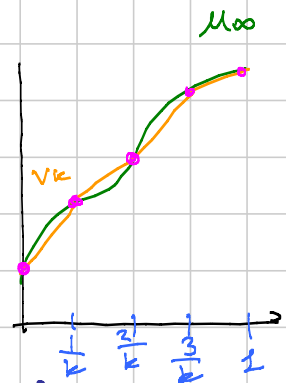
We know that

$$u_n\left(\frac{i}{k}\right) \rightarrow v_k\left(\frac{i}{k}\right) \text{ as } n \rightarrow \infty$$

and therefore

$$\liminf_{n \rightarrow \infty} F_n(u_n) \geq \sum_{i=1}^k \frac{2\alpha\beta}{\alpha+\beta} \left[v_k\left(\frac{i}{k}\right) - v_k\left(\frac{i-1}{k}\right) \right]^2 k$$

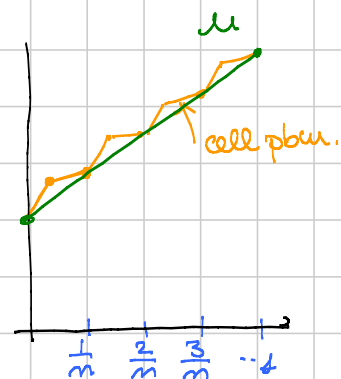
$$= \frac{2\alpha\beta}{\alpha+\beta} \int_0^1 \dot{v}_k^2 = F_\infty(v_k)$$



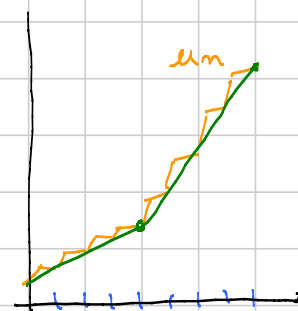
This is true for every k . But the sup of the RHS w.r.t k is exactly $F_\infty(u_\infty)$.

linsup req.

→ Easy case: u is a line
 $\leadsto$ just use the solution to the
 cell problem repeated



→ Medium case: u is piecewise
 affine: $\leadsto$ approx each line
 as before



→ General case: any $u \in L^2$
 $\leadsto$ just use that piecewise
 affine functions are dense
 in energy w.r.t $F_\alpha(u)$.

— o — o —

Calculus of Variations — LECTURE 48

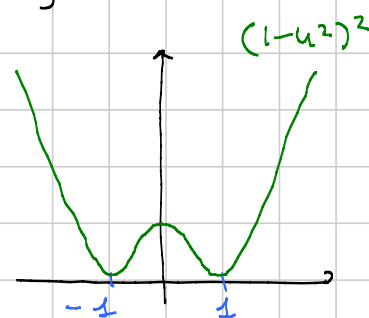
Note Title

21/12/2016

MODICA - MORTOLA

$$\min \left\{ \int_0^1 \varepsilon^2 \dot{u}^2 + \underbrace{(1-u^2)^2}_{\cos u} dx : u(0)=0 \right\} = m_\varepsilon$$

we expect u to be ± 1 in the limit



Since the pbm. is symmetric, we can add the constraint $u \geq 0$.

① m_ε is well defined, namely the pbm. has a solution

Proof: standard direct method.

② m_ε , as stated, tends to 0, because the T-Limit is

$$\int_0^1 (1-u^2)^2 dx$$

③ we need to rescale

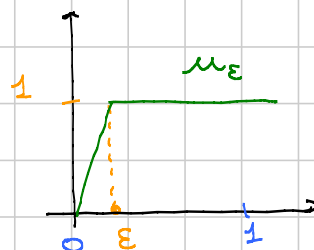
$$F_\varepsilon(u) = \int_0^1 \varepsilon \dot{u}^2 + \frac{1}{\varepsilon} (1-u^2)^2$$

Modica - Mortola
functional

$$\min \{ F_\varepsilon(u) : u(0)=0, u(x) \geq 0 \} \leq M \text{ indep. of } \varepsilon$$

Proof.

$$\begin{aligned} \min &\leq F_\varepsilon(u_\varepsilon) = \int_0^\varepsilon \varepsilon \cdot \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} (1 - u_\varepsilon^2)^2 \\ &\leq 1 + 1 = 2 = M \end{aligned}$$

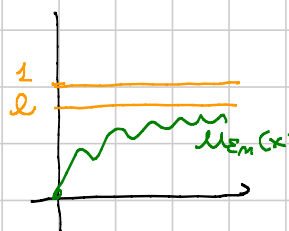


③ let us assume that $u_\varepsilon \in \arg\min F_\varepsilon$. Then

$$\liminf_{\varepsilon \rightarrow 0^+} \max_{x \in [0,1]} u_\varepsilon \geq 1 \quad [\text{actually } \max u_\varepsilon \rightarrow 1]$$

Proof. Assume by contradiction that $\liminf < 1$.
This means that there exist $\ell < 1$
and $\varepsilon_m \rightarrow 0$ such that

$$u_{\varepsilon_m}(x) \leq \ell \quad \forall x \in [0,1], \forall m \in \mathbb{N}$$



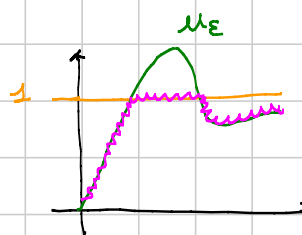
But in this case

$$\begin{aligned} u_{\varepsilon_m} = F_{\varepsilon_m}(u_{\varepsilon_m}) &\geq \int_0^1 \frac{1}{\varepsilon_m} (1 - u_{\varepsilon_m}^2)^2 dx \\ &\geq \frac{1}{\varepsilon_m} \int_0^1 (1 - \ell^2)^2 dx \rightarrow +\infty \text{ as } m \rightarrow +\infty \end{aligned}$$

Remark A standard truncation argument shows that actually

$$0 \leq u_\varepsilon(x) \leq 1 \quad \forall x \in [0,1]$$

if $u_\varepsilon \in \arg\min F_\varepsilon$.



This proves that $\max_{x \in [0,1]} u_\varepsilon \rightarrow 1$.

④ Let us consider the function

$$G(x) := \int_0^x |1-t^2| dt$$

namely the function such that $G(0) = 0$ and

$$G'(x) = |1-x^2| \quad \forall x \in \mathbb{R}.$$

We claim that

$$\liminf_{\varepsilon \rightarrow 0} \min \{ F_\varepsilon(u) : u(0) = 0, u(x) \geq 0 \} \geq 2 G(1) > 0$$

Proof. Let u be any competitor. Then

$$\begin{aligned} F_\varepsilon(u) &= \int_0^1 \left[\varepsilon u^2 + \frac{1}{\varepsilon} (1-u^2)^2 \right] dx \\ &\geq 2 \int_0^1 |u(1-u^2)| dx \\ &= 2 \int_0^1 | [G(u)]' | dx \\ &\geq 2 \left| \int_0^1 [G(u)]' dx \right| \\ &= 2 | G(u(1)) - G(u(0)) | \end{aligned}$$

$A^2 + B^2 \geq 2AB$
 $A = \sqrt{\varepsilon} |u|$
 $B = \frac{1}{\sqrt{\varepsilon}} |1-u^2|$

In the same way, let $u_\varepsilon \in \arg\min F_\varepsilon$ and let a_ε be a max point of $u_\varepsilon(x)$ in $[0, 1]$. Then

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &= \int_0^1 \dots \geq \int_0^{a_\varepsilon} \dots \geq 2 | G(u(a_\varepsilon)) - \underbrace{G(u(0))}_0 | \\ &= 2 | G(\underbrace{\max u_\varepsilon}_{\downarrow 1}) | \end{aligned}$$

So in the limit $\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq 2 G(1).$

⑤ Actually it turns out that

$$\lim_{\varepsilon \rightarrow 0^+} \min \{ F_\varepsilon(u) : u(0) = 0, u(x) \geq 0 \} = 2G(1)$$

Proof. we need to find u_ε s.t.

$$\limsup_{\varepsilon \rightarrow 0^+} F_\varepsilon(u_\varepsilon) \leq 2G(1)$$

In $A^2 + B^2 \geq 2AB$ we have equality when $A = B$, namely

$$\sqrt{\varepsilon} |\dot{u}(x)| = \frac{1}{\sqrt{\varepsilon}} |1 - u^2(x)|$$

namely $\dot{u}(x) = \frac{1}{\varepsilon} (1 - u^2(x))$

$$\frac{\dot{u}}{1 - u^2} = \frac{1}{\varepsilon}$$

and the solution is

$$u(x) = \tanh\left(\frac{x}{\varepsilon} + k\right)$$

$\uparrow$
 $k=0$ because $u(0)=0$

If we take $u_\varepsilon(x) = \tanh\left(\frac{1}{\varepsilon}x\right)$, then we obtain

$$F_\varepsilon(u_\varepsilon) = 2G\left(\tanh\left(\frac{1}{\varepsilon}\right)\right) \rightarrow 2G(1)$$



Remark u_ε is not the min. point, because it does not fulfil NBC in $x=1$, but it is "almost" a min point.

— 0 — 0 —

⑥ In some sense, the $\tanh$ is the unique "transition profile", in the following sense: see later

⑤.5) Let $u_\varepsilon \in \arg\min F_\varepsilon$. Then for every $a \in (0, 1)$

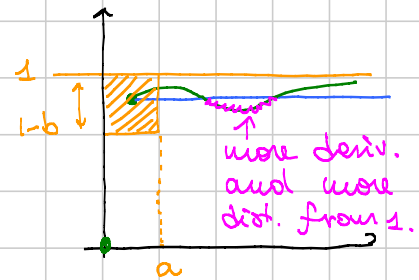
$$u_\varepsilon \rightarrow 1 \text{ uniformly in } [a, 1]$$

Proof. For ε small enough u_ε has a point in the square $(0, a) \times (1-b, 1)$

If not

$$F_\varepsilon(u_\varepsilon) \geq \int_0^a \frac{1}{\varepsilon} (1-u^2)^2 \geq \int_0^a \frac{1}{\varepsilon} b^2 \rightarrow +\infty$$

Finally, u_ε is increasing



⑥ If $u_\varepsilon \in \arg\min$, then $u_\varepsilon(\varepsilon x) \rightarrow \tanh x$

Idea: Let us set $v_\varepsilon(x) := u_\varepsilon(\varepsilon x)$. Note that

$$v_\varepsilon: [0, \frac{1}{\varepsilon}] \rightarrow \mathbb{R}$$

and v_ε is the solution to the min. pbm

$$\min \left\{ \int_0^{\frac{1}{\varepsilon}} \dot{v}^2 + (1-v^2)^2 dx : \begin{array}{l} v(0) = 0 \\ v(x) \geq 0 \end{array} \right\}$$

and it is quite standard to see that this pbm Γ -converges to

$$\min \left\{ \int_0^{+\infty} \dot{v}^2 + (1-v^2)^2 dx : v(0) = 0, v(x) \geq 0 \right\}$$

and the solution to this pbm. is $v(x) = \tanh x$.

Final remark what is the Γ -limit of F_ε ?

Let us denote it by $F_0(u)$. Then

$$F_0(u) < +\infty \Rightarrow u(x) = \pm 1 \text{ for almost every } x \in [0,1]$$

↑
easy fact

Fact: $F_0(u) < +\infty \Leftrightarrow u(x) = \pm 1$ for a.e. $x \in [0,1]$
and there is a finite number
of transitions between ± 1

and in this case

$$F_0(u) = 4G(u) \cdot \text{number of transitions}$$



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