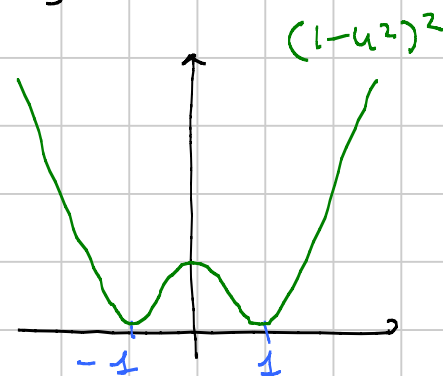
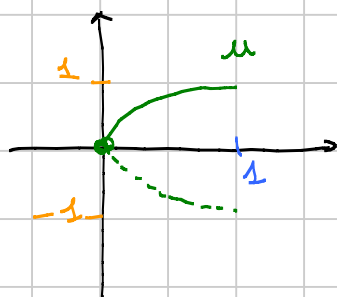


MODICA - MORTOLA

$$\min \left\{ \int_0^1 \varepsilon^2 \dot{u}^2 + \underbrace{(1-u^2)^2}_{\cos u} dx : u(0)=0 \right\} = m_\varepsilon$$

we expect u to be ± 1 in the limit



Since the pbm. is symmetric, we can add the constraint $u \geq 0$.

① m_ε is well defined, namely the pbm. has a solution

Proof: standard direct method.

② m_ε , as stated, tends to 0, because the T-limit is

$$\int_0^1 (1-u^2)^2 dx$$

③ we need to rescale

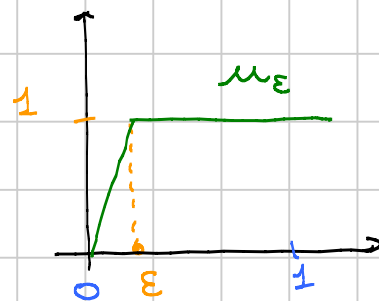
$$F_\varepsilon(u) = \int_0^1 \varepsilon \dot{u}^2 + \frac{1}{\varepsilon} (1-u^2)^2$$

Modica - Mortola
functional

$$\min \{ F_\varepsilon(u) : u(0)=0, u(x) \geq 0 \} \leq M \text{ indep. of } \varepsilon$$

Proof.

$$\begin{aligned} \min &\leq F_\varepsilon(u_\varepsilon) = \int_0^\varepsilon \varepsilon \cdot \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} (1-u_\varepsilon^2)^2 \\ &\leq 1+1 = 2 = \gamma \end{aligned}$$

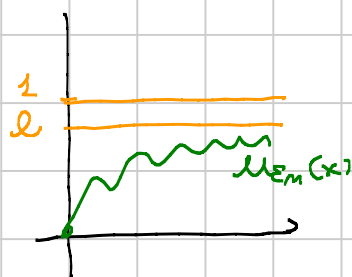


③ Let us assume that $u_\varepsilon \in \arg\min F_\varepsilon$. Then

$$\liminf_{\varepsilon \rightarrow 0^+} \max_{x \in [0,1]} u_\varepsilon \geq 1 \quad [\text{actually } \max u_\varepsilon \rightarrow 1]$$

Proof. Assume by contradiction that $\liminf < 1$.
This means that there exist $\ell < 1$
and $\varepsilon_n \rightarrow 0$ such that

$$u_{\varepsilon_n}(x) \leq \ell \quad \forall x \in [0,1], \forall n \in \mathbb{N}$$



But in this case

$$\begin{aligned} u_{\varepsilon_n} = F_{\varepsilon_n}(u_{\varepsilon_n}) &\geq \int_0^1 \frac{1}{\varepsilon_n} (1-u_{\varepsilon_n}^2)^2 dx \\ &\geq \frac{1}{\varepsilon_n} \int_0^1 (1-\ell^2)^2 dx \rightarrow +\infty \text{ as } n \rightarrow +\infty \end{aligned}$$

Remark A standard truncation argument shows that actually

$$0 \leq u_\varepsilon(x) \leq 1 \quad \forall x \in [0,1]$$

If $u_\varepsilon \in \arg\min F_\varepsilon$.



This proves that $\max_{x \in [0,1]} u_\varepsilon \rightarrow 1$.

④ Let us consider the function

$$G(x) := \int_0^x |1-t^2| dt$$

namely the function such that $G(0) = 0$ and

$$G'(x) = |1-x^2| \quad \forall x \in \mathbb{R}.$$

We claim that

$$\liminf_{\varepsilon \rightarrow 0} \min \{ F_\varepsilon(u) : u(0) = 0, u(x) \geq 0 \} \geq 2 G(1) > 0$$

Proof. Let u be any competitor. Then

$$F_\varepsilon(u) = \int_0^1 \left[\varepsilon \dot{u}^2 + \frac{1}{\varepsilon} (1-u^2)^2 \right] dx$$

$$A^2 + B^2 \geq 2AB$$

$$A = \sqrt{\varepsilon} |\dot{u}|$$

$$B = \frac{1}{\sqrt{\varepsilon}} |1-u^2|$$

$$\geq 2 \int_0^1 |\dot{u}| (1-u^2) dx$$

$$= 2 \int_0^1 | [G(u)]' | dx$$

$$\geq 2 \left| \int_0^1 [G(u)]' dx \right|$$

$$= 2 | G(u(1)) - G(u(0)) |$$

In the same way, let $u_\varepsilon \in \arg\min F_\varepsilon$ and let a_ε be a max point of $u_\varepsilon(x)$ in $[0, 1]$. Then

$$F_\varepsilon(u_\varepsilon) = \int_0^1 \dots \geq \int_0^{a_\varepsilon} \dots \geq 2 \left| G(u(a_\varepsilon)) - \underbrace{G(u(0))}_0 \right|$$

$$= 2 \left| G(\underbrace{\max_{x \in [0,1]} u_\varepsilon}_1) \right|$$

So in the limit $\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq 2 G(1).$

⑤ Actually it turns out that

$$\lim_{\varepsilon \rightarrow 0^+} \min \{ F_\varepsilon(u) : u(0) = 0, u(x) \geq 0 \} = 2G(1)$$

Proof, we need to find u_ε s.t.

$$\limsup_{\varepsilon \rightarrow 0^+} F_\varepsilon(u_\varepsilon) \leq 2G(1)$$

In $A^2 + B^2 \geq 2AB$ we have equality when $A = B$, namely

$$\sqrt{\varepsilon} | \ddot{u}(x) | = \frac{1}{\sqrt{\varepsilon}} | 1 - u^2(x) |$$

$$\text{namely} \quad \ddot{u}(x) = \frac{1}{\varepsilon} (1 - u^2(x)) \quad \frac{\ddot{u}}{1 - u^2} = \frac{1}{\varepsilon}$$

and the solution is

$$u(x) = \tanh\left(\frac{x}{\varepsilon} + k\right)$$

$\uparrow$
 $k=0$ because $u(0)=0$

If we take $u_\varepsilon(x) = \tanh\left(\frac{1}{\varepsilon}x\right)$, then we obtain

$$F_\varepsilon(u_\varepsilon) = 2G\left(\tanh\left(\frac{1}{\varepsilon}\right)\right) \rightarrow 2G(1)$$

Remark u_ε is not the min. point, because it does not fulfil NBC in $x=1$, but it is "almost" a min point.

— 0 — 0 —



⑥ In some sense, the tanh is the unique "transition profile", in the following sense: see later

⑤.5) Let $u_\varepsilon \in \arg\min F_\varepsilon$. Then for every $a \in (0, 1)$

$$u_\varepsilon \rightarrow 1 \text{ uniformly in } [a, 1]$$

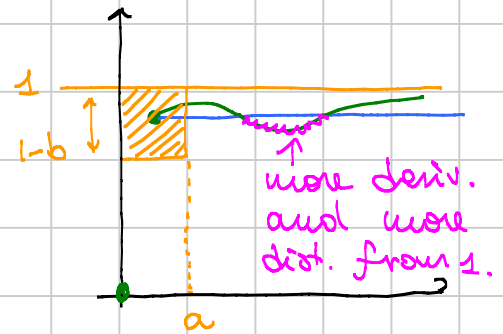
Proof. For ε small enough u_ε has a point in the square

$$(0, a) \times (1-b, 1)$$

If not

$$F_\varepsilon(u_\varepsilon) \geq \int_0^a \frac{1}{\varepsilon} (1-u^2)^2 \geq \int_0^a \frac{1}{\varepsilon} b^2 \rightarrow +\infty$$

Finally, u_ε is increasing



⑥ If $u_\varepsilon \in \arg\min$, then $u_\varepsilon(\varepsilon x) \rightarrow \tanh x$

Idea: Let us set $v_\varepsilon(x) := u_\varepsilon(\varepsilon x)$. Note that

$$v_\varepsilon: [0, \frac{1}{\varepsilon}] \rightarrow \mathbb{R}$$

and v_ε is the solution to the min. pbm

$$\min \left\{ \int_0^{\frac{1}{\varepsilon}} \dot{v}^2 + (1-v^2)^2 dx : \begin{array}{l} v(0) = 0 \\ v(x) \geq 0 \end{array} \right\}$$

and it is quite standard to see that this pbm Γ -converges to

$$\min \left\{ \int_0^{+\infty} \dot{v}^2 + (1-v^2)^2 dx : v(0) = 0, v(x) \geq 0 \right\}$$

and the solution to this pbm. is $v(x) = \tanh x$.

Final remark What is the Γ -limit of F_ε ?

Let us denote it by $F_0(u)$. Then

$$F_0(u) < +\infty \Rightarrow u(x) = \pm 1 \text{ for almost every } x \in [0,1]$$

↑
easy fact

Fact: $F_0(u) < +\infty \Leftrightarrow u(x) = \pm 1$ for a.e. $x \in [0,1]$
and there is a finite number
of transitions between ± 1

and in this case

$$F_0(u) = 4G(1) \cdot \text{number of transitions}$$

— 0 — 0 —

