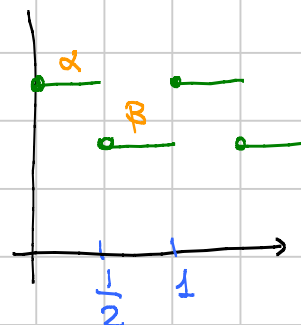


HOMOGENIZATION PROBLEMSGreek function $c: [0, 1] \rightarrow \mathbb{R}$

$$c(x) = \begin{cases} \alpha & \text{if } x \in [0, \frac{1}{2}) \\ \beta & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

Let us extend to $c: \mathbb{R} \rightarrow \mathbb{R}$ by periodicity.Let us set $c_n(x) := c(nx)$ so that c_n is $\frac{1}{n}$ -periodic

Exercise $c_n \rightarrow \frac{\alpha+\beta}{2}$ in $L^p(a, b)$
for every $p \geq 1$ ($p \neq \infty$) and
for every $(a, b) \subseteq \mathbb{R}$.



Functionals: $G_n(u) := \int_0^1 \dot{u}^2 + c_n(x) u^2$ $\leadsto$ easy case
 $F_n(u) := \int_0^1 c_n(x) \dot{u}^2 + u^2$ $\leadsto$ diff. case

Theorem 1 $G_n(u) \xrightarrow{\Gamma} G_\infty(u)$, where

$$G_\infty(u) = \int_0^1 \dot{u}^2 + \frac{\alpha+\beta}{2} u^2$$

More generally, if $G_n(u) = \int_0^1 \dot{u}^2 + \underbrace{f_n(x)}_{\geq 0} u^2$ and $f_n \rightarrow f_\infty$ weakly, then

$$G_n(u) \rightarrow \int_0^1 \dot{u}^2 + f_\infty(x) u^2$$

Proof. Liminf ineq. Let $u_n \rightarrow u_\infty$ in L^2 (or C^0) and let us assume that

$$\liminf G_n(u_n) < +\infty$$

and actually a limit. Since $f_n(x) \geq 0$, we have a bound on $\|u_n\|_{L^2}$ and therefore (up to subsequences)

$$u_n \rightarrow u_\infty \quad u_n \rightarrow u_\infty \text{ unif.}$$

We claim that $\liminf_{n \rightarrow +\infty} \int_0^1 u_n^2 \geq \int_0^1 u_\infty^2$ (classical)

$$\lim_{n \rightarrow +\infty} \int_0^1 f_n(x) u_n^2 = \int_0^1 f_\infty(x) u_\infty^2.$$

$$f_n \rightarrow f_\infty, \quad u_n \rightarrow u_\infty \text{ unif.} \Rightarrow u_n^2 \rightarrow u_\infty^2 \text{ unif.}$$

$$\Rightarrow u_n^2 \rightarrow u_\infty^2 \text{ in } L^2 \text{ strong.}$$

$\int_0^1 f_n \cdot u_n^2$ is the scalar prod. of a strongly conv. and a weakly conv. sequence.

Limsup ineq. For every $u \in L^2$ we used a recovery sequence. It is enough if $G(u) < +\infty$, and therefore $u \in H^1$, and therefore $u \in C^0$.

Now we can use $u_n \equiv u$, and trivially

$$\int_0^1 f_n(x) u^2 \rightarrow \int_0^1 f_\infty(x) u^2.$$

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Theorem 2 Assume that $\alpha > 0, \beta > 0$. Then

$$F_m(u) \xrightarrow{\Gamma} F_\infty(u) = \int_0^1 \frac{2\alpha\beta}{\alpha+\beta} \dot{u}^2 + u^2$$

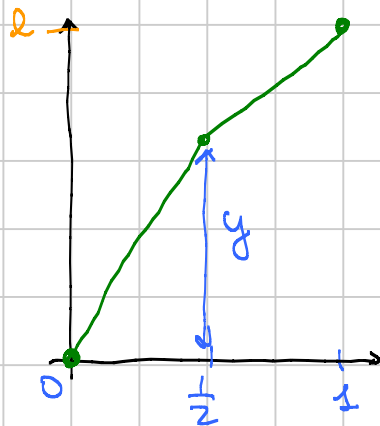
↑
Harmonic mean of α and β

[It is enough to consider $F_m(u) := \int_0^1 c_m(x) \dot{u}^2$]

CELL PROBLEM

$$\min \left\{ \int_0^1 c(x) \dot{u}^2 dx : u(0) = 0, u(1) = l \right\}$$
$$\int_0^{\frac{1}{2}} \alpha \dot{u}^2 + \int_{\frac{1}{2}}^1 \beta \dot{u}^2$$

we can restrict to functions as here $\rightarrow$



$$\frac{1}{2} \alpha (2y)^2 + \frac{1}{2} \beta (2(l-y))^2$$

$$= 2 [\alpha y^2 + \beta (l-y)^2] \rightsquigarrow \min \text{ wrt } y$$

$$\alpha y - \beta (l-y) = 0 \Leftrightarrow y(\alpha+\beta) = \beta l \Leftrightarrow y = \frac{\beta}{\alpha+\beta} l$$

Replacing this value, the min is

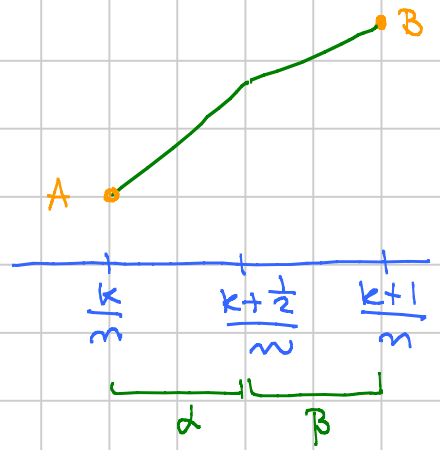
$$2 \left[\alpha \frac{\beta^2}{(\alpha+\beta)^2} l^2 + \beta \frac{\alpha^2}{(\alpha+\beta)^2} l^2 \right] = 2 \frac{\alpha\beta}{(\alpha+\beta)^2} l^2 (\alpha+\beta)$$
$$= \frac{2\alpha\beta}{\alpha+\beta} l^2$$

Conclusion The minimum of the cell problem is the same as

$$\min \left\{ \int_0^1 \frac{2\alpha\beta}{\alpha+\beta} \dot{u}^2 dx : u(0) = 0, u(1) = l \right\}$$

Modified cell pbm

$$\min \left\{ \int_{\frac{k}{n}}^{\frac{k+1}{n}} C_n(x) \dot{u}^2 : u\left(\frac{k}{n}\right) = A \right. \\ \left. u\left(\frac{k+1}{n}\right) = B \right\}$$



As before the min is

$$\frac{2\alpha\beta}{\alpha+\beta} (B-A)^2 n$$

$$\left[n(B-A) \right]^2 \cdot \frac{1}{n}$$

$\uparrow$ derivative of the line $\uparrow$ interval of integration

Special case let us assume that $u_n \rightarrow u_\infty$ unif. where $u_\infty(x) = 2x$ on $[0, 1]$.

Then

$$\liminf_{n \rightarrow \infty} F_n(u_n) \geq F_\infty(u_\infty)$$

Proof.

$$F_n(u_n) = \int_0^1 C_n(x) \dot{u}_n^2 dx$$

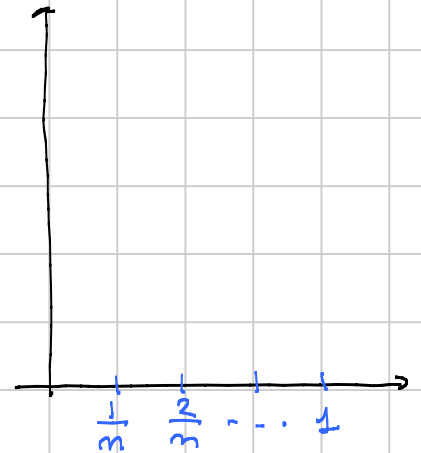
$$= \sum_{k=1}^n \underbrace{\int_{\frac{k-1}{n}}^{\frac{k}{n}} C_n(x) \dot{u}_n^2 dx}_{\text{modified cell pbm}}$$

$$\geq \sum_{k=1}^n \left(\frac{2\alpha\beta}{\alpha+\beta} \right) n \left[u_n\left(\frac{k}{n}\right) - u_n\left(\frac{k-1}{n}\right) \right]^2$$

$$n \sum \alpha_k^2 \geq (\sum \alpha_k)^2$$

$$\stackrel{\text{C.S.}}{\geq} \frac{2\alpha\beta}{\alpha+\beta} \left(\sum_{k=1}^n u_n\left(\frac{k}{n}\right) - u_n\left(\frac{k-1}{n}\right) \right)^2$$

$$= \frac{2\alpha\beta}{\alpha+\beta} [u_n(1) - u_n(0)]^2 = \frac{2\alpha\beta}{\alpha+\beta} 2^2 = F_\infty(u_\infty)$$



Analogous case Let $(a, b) \subseteq \mathbb{R}$ be any interval.

Assume that

$$u_n \rightarrow u_\infty(x) \text{ unif. in } [a, b].$$

$$\text{Then } \liminf_{n \rightarrow \infty} F_n(u_n) \geq \frac{2\alpha\beta}{\alpha+\beta} [u_\infty(b) - u_\infty(a)]^2 \frac{1}{b-a}$$

If $u_\infty(x)$ is a line, then the RHS is actually $F_\infty(u_\infty)$

Proof : more or less the same as above.
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Proof that $F_n \xrightarrow{\Gamma} F_\infty$ always

liminf ineq. Let us fix $k \geq 1$. Let $u_n \rightarrow u_\infty$ in L^2 with $\liminf F_n(u_n) < +\infty$.

As always we obtain that $u_n \rightarrow u_\infty$ unif.

Let v_k be the piecewise approx of u_∞ with k subintervals.

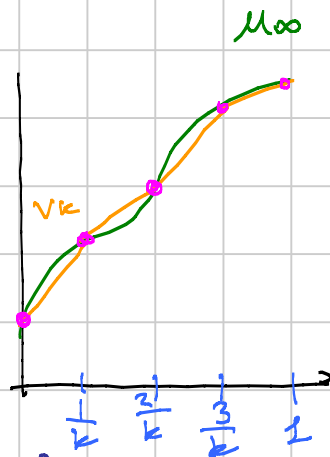
We know that

$$u_n\left(\frac{i}{k}\right) \rightarrow v_k\left(\frac{i}{k}\right) \text{ as } n \rightarrow \infty$$

and therefore

$$\liminf_{n \rightarrow \infty} F_n(u_n) \geq \sum_{i=1}^k \frac{2\alpha\beta}{\alpha+\beta} \left[v_k\left(\frac{i}{k}\right) - v_k\left(\frac{i-1}{k}\right) \right]^2 k$$

$$= \frac{2\alpha\beta}{\alpha+\beta} \int_0^1 \dot{v}_k^2 = F_\infty(v_k)$$



This is true for every k . But the sup of the RHS wrt k is exactly $F_\infty(u_\infty)$.

limsup ineq.

→ Easy case: u is a line
→ just use the solution to the cell problem repeated

→ Medium case: u is piecewise affine:
→ approx each line as before

→ General case: any $u \in L^2$
→ just use that piecewise affine functions are dense in energy wrt $F_\alpha(u)$.

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